5.1

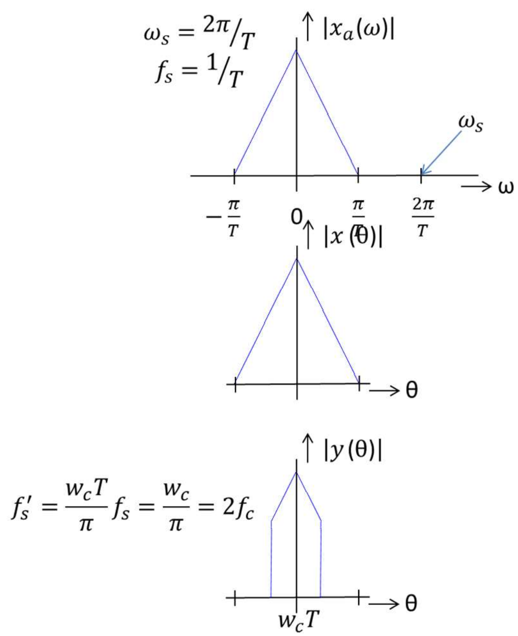
Change of sampling rate

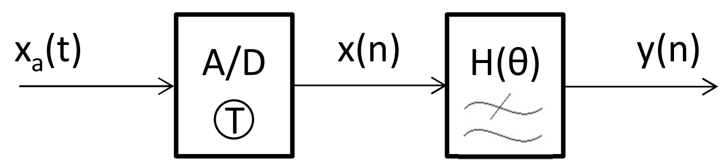
1. Why
2. Decrease
3. Increase
4. Transposition

Rabiner Croctier
Multirate Signal Processing

1. Why change of sampling rate

Example decrease



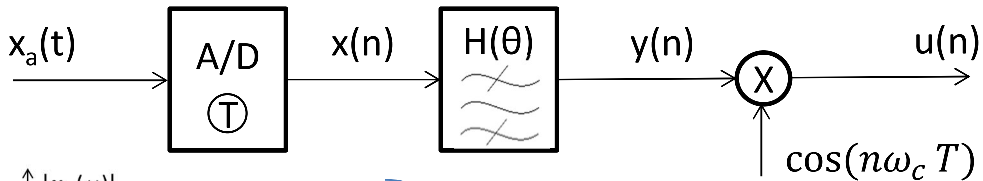


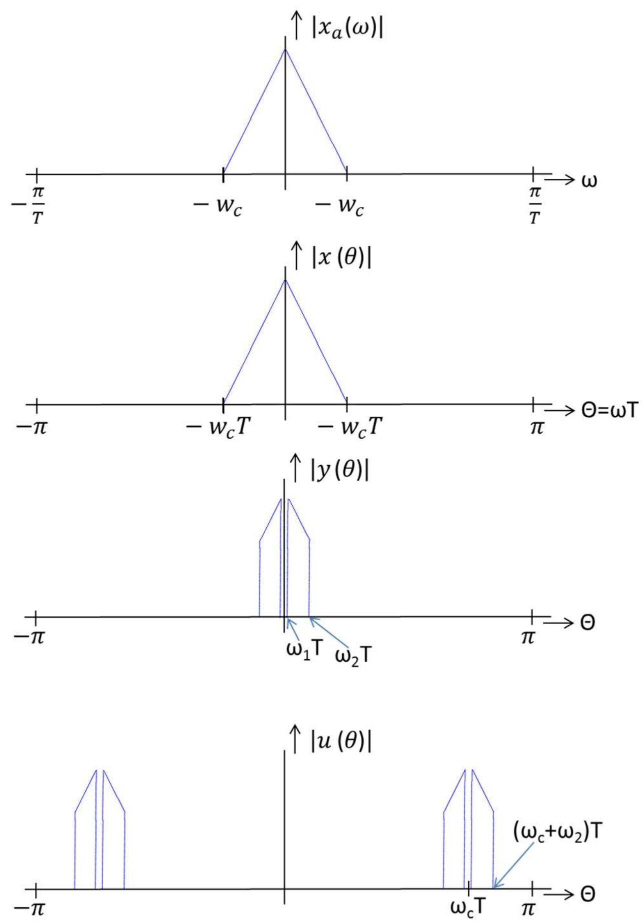
If $2f_c \ll f_s$ it is advantageous to lower f_s

20/04/2020

5.2

Example increase





$f_s' \leq f_s$

$(\omega_c+\omega_2)T<\pi$

$$f_s = \frac{1}{T} > \frac{\omega_c+\omega_2}{\pi}$$

$$Or: \quad f_s > 2(f_c + f_2)$$

Introduce sample rate decrease (SDR) and sample rate increase (SRI)

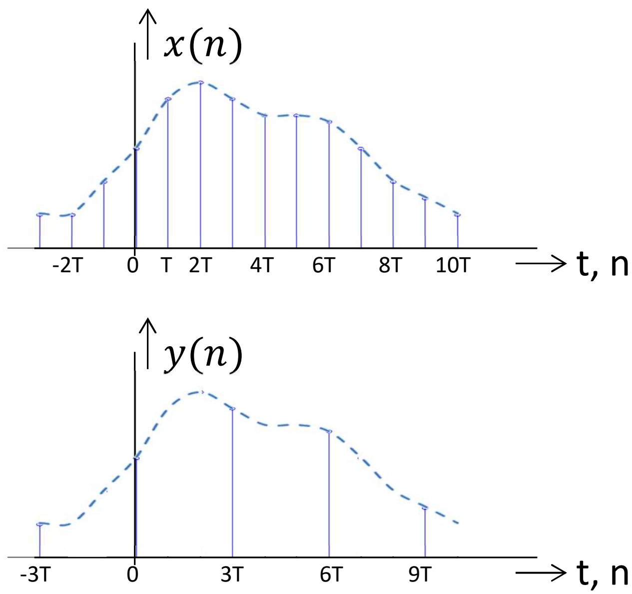
In actual systems both SRD and SRI can hardly be recognized as separate entities

2

5.3

2. Sampling rate decrese (SRD)

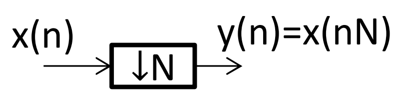
Reduction factor N=3



Notation:

$$y(n) = x(nN) \quad with\, n = 0, \pm 1, \pm 2, \ldots$$

Symbol of an SRD:



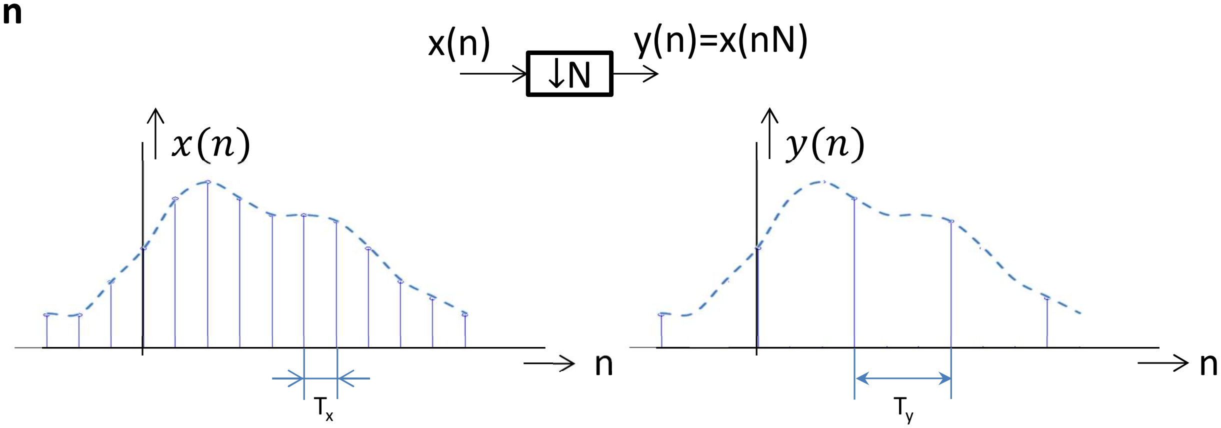
Question: Is it possible to describe an SRD by it's impulse response; it's transmission function or it's system function? NO

It is an invariant time system

3

Notation

5.4



	Until now	x(n)	y(n)
Sampling period	T	T_x	T_y
Normalized frequency	$\theta = \omega T$	$\theta_x = \omega T_x$	$\theta_y = \omega T_y$
Fundamental interval	$\lvert\theta\rvert \leq \pi$ $\lvert\omega\rvert \leq \pi/T$	$\lvert\theta_x\rvert \leq \pi$ $\lvert\omega_x\rvert \leq \pi/T_x$	$\lvert\theta_y\rvert \leq \pi$ $\lvert\omega_y\rvert \leq \pi/T_y$

Relations:

$$T_y = NT_x$$

$$\theta_y = \omega T_y = \omega N T_x = N\theta_x$$

Relation between spectra of x(n) and y(n)

5.5

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta_x)e^{jn\theta_x}d\theta_x = \frac{1}{2\pi}\int_{0}^{2\pi} X(\theta_x)e^{jn\theta_x}d\theta_x$$

$$x(nN) = \frac{1}{2\pi}\int_{0}^{2\pi} X(\theta_x)e^{jnN\theta_x}d\theta_x$$

Substitute: $\theta_y = N\theta_x$

$$x(nN) = \frac{1}{2\pi}\int_{0}^{2\pi} X\left(\frac{\theta_y}{N}\right)e^{jn\theta_y}\frac{d\theta_y}{N} = \frac{1}{2\pi N}\sum_{k=0}^{N-1}\int_{k2}^{(k+1)2\pi} X\left(\frac{\theta_y}{N}\right)e^{jn\ y}d\theta_y \qquad \theta_y \to \theta_y + 2k\pi$$

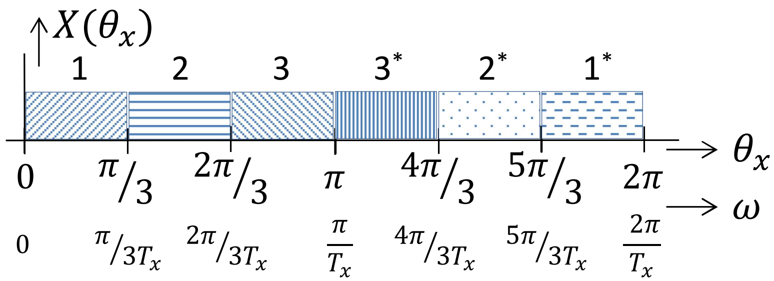
$$= \frac{1}{2\pi N}\sum_{k=0}^{N-1}\int_{0}^{2\pi} X\left(\frac{\theta_y + 2\pi k}{N}\right)e^{jn(\theta_y+2\pi k)}d\theta_y$$

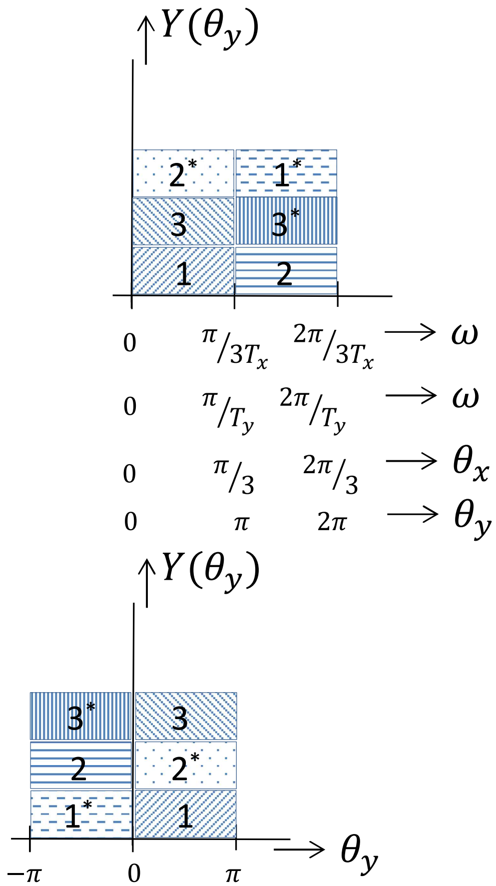
$$x(nN) = \frac{1}{2\pi}\int_{0}^{2\pi}\frac{1}{N}\sum_{k=0}^{N-1} X\left(\theta_x + \frac{2\pi k}{N}\right)e^{jn\theta_y}\,d\theta_y$$

$$x(nN) = y(n) = \frac{1}{2\pi}\int_{0}^{2\pi} Y(\theta_y)e^{jn\ y}d\theta_y$$

$$\boxed{Y(\theta_y) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\theta_x + \frac{2\pi k}{N}\right)}$$

5.6





$$Y(\theta_y) = \frac{1}{3}\sum_{k=0}^{2} X(\theta_x + {}^{2k\pi}/_{3})$$

$$= \frac{1}{3}\left[X(\theta_x) + X(\theta_x + {}^{2\pi}/_{3}) + X(\theta_x + {}^{4\pi}/_{3})\right]$$

$$\theta_y = 3\theta_x$$

Terms 2, 2*, 3 and 3* are aliasing terms.

No aliasing if

$$X(\theta_x) = 0 \quad for \quad \frac{\pi}{N} \le \theta_x \le \pi$$

Then

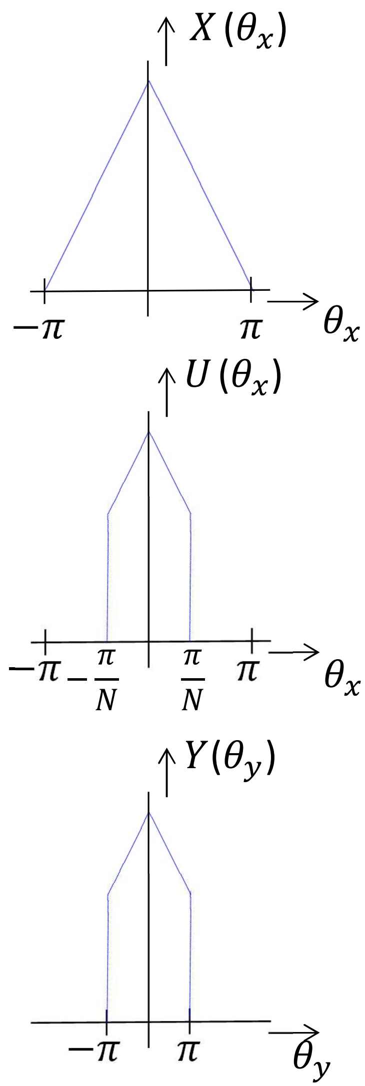
$$Y(\theta_y) = \frac{1}{N} X(\theta_x) = \frac{1}{N} X\left(\frac{\theta_y}{N}\right) \qquad -\pi \le \theta_y \le \pi$$

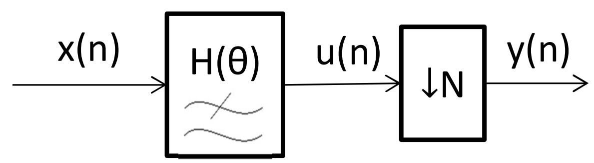
Or

$$Y(\omega T_y) = \frac{1}{N} X(\omega T_x) \qquad -{}^{\pi}/_{T_y} \le \omega \le {}^{\pi}/_{T_y}$$

Decimator: Ideal low pass + SRD

5.7



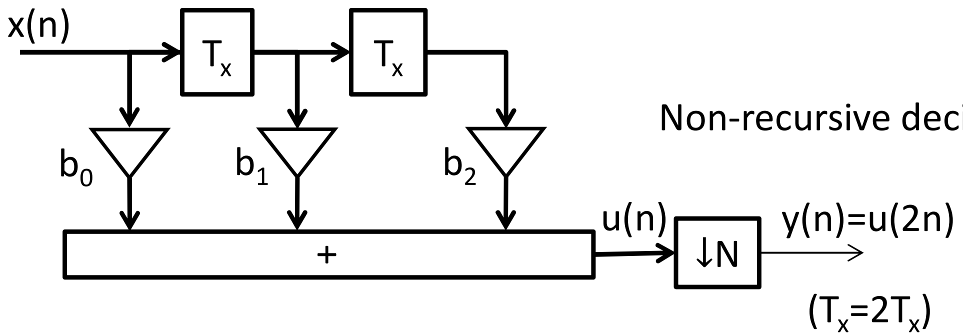


Decimator: Removes unwanted spectrum and induces no aliasing

Remark: The ideal low pass filter does not exist; in practice actual low pass in cascade with an SRD (decimating filter)

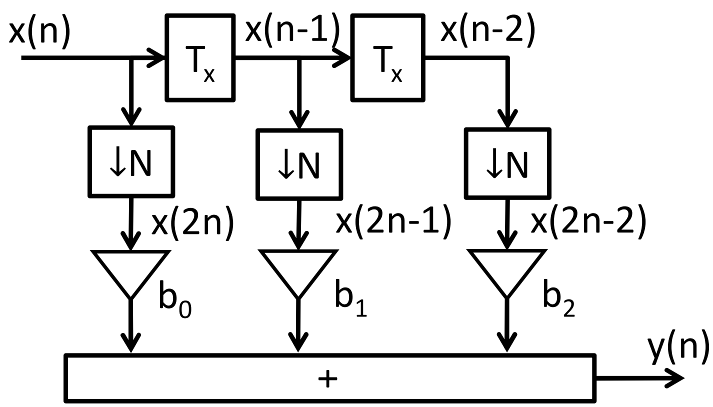
5.8

Implementation aspects



Non-recursive decimating filter

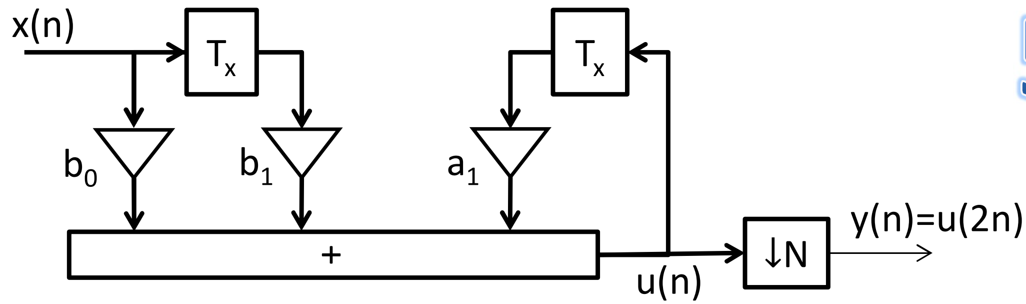
$$y(0) = u(0) = b_0 x(0) + b_1 x(-1) + b_2 x(-2)$$
$$y(1) = u(2) = b_0 x(2) + b_1 x(1) + b_2 x(0)$$
$$y(2) = u(4) = b_0 x(4) + b_1 x(3) + b_2 x(2)$$
$$\vdots$$
$$y(n) = u(2n) = b_0 x(2n) + b_1 x(2n-1) + b_2 x(2n-2)$$



Less multiplications (50%)

Recursive filter + SRD

5.9



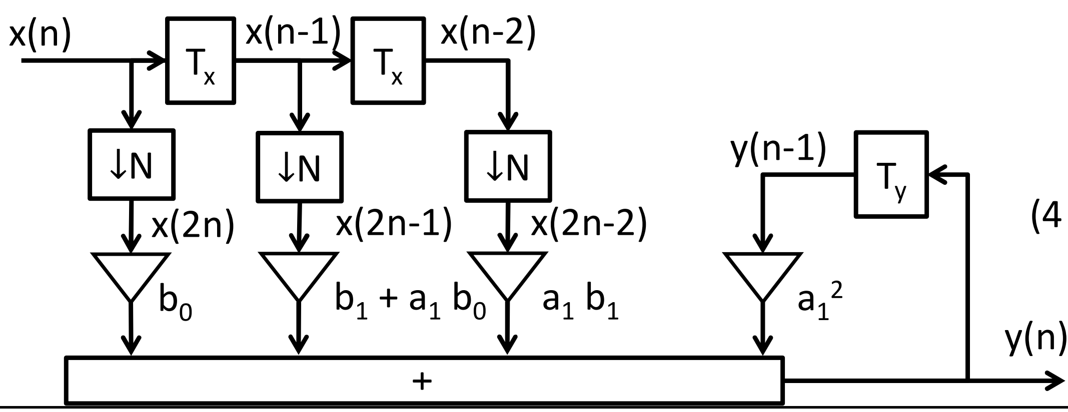
$$u(n) = b_0x(n) + b_1x(n-1) + a_1u(n-1) \quad (1)$$

$$So: \quad u(n-1) = b_0x(n-1) + b_1x(n-2) + a_1u(n-2) \quad (2)$$

Combine (1) and (2)

$$u(n) = b_0x(n) + b_1x(n-1) + a_1\big(b_0x(n-1) + b_1x(n-2) + a_1u(n-2)\big)$$
$$= b_0x(n) + (b_1+a_1b_0)x(n-1) + a_1b_1x(n-2) + a_1{}^2u(n-2)$$
$$\Downarrow$$
$$u(2n) = b_0x(2n) + (b_1+a_1b_0)x(2n-1) + a_1b_1x(2n-2) + a_1{}^2u(2n-2)$$

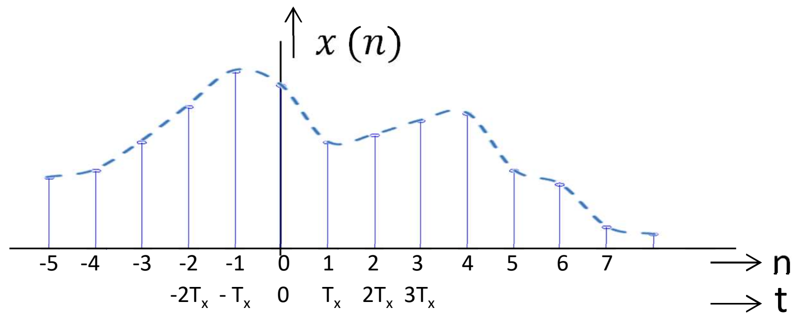
$$y(n) = b_0x(2n) + (b_1+a_1b_0)x(2n-1) + a_1b_1x(2n-2) + a_1{}^2y(n-1)$$

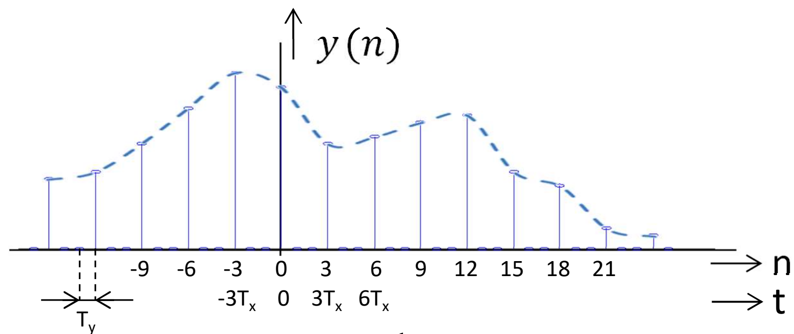


Less multiplications
(4 instead of 6 per T_y seconds)

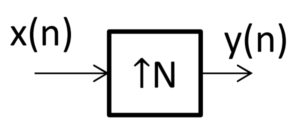
Sampling rate increase (SRI)

5.10



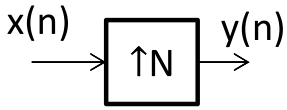


$$y(n) = \begin{cases} x\left(\frac{n}{N}\right) & for \quad n = 0, \pm N, \pm 2N, \ldots \\ 0 & otherwise \end{cases}$$

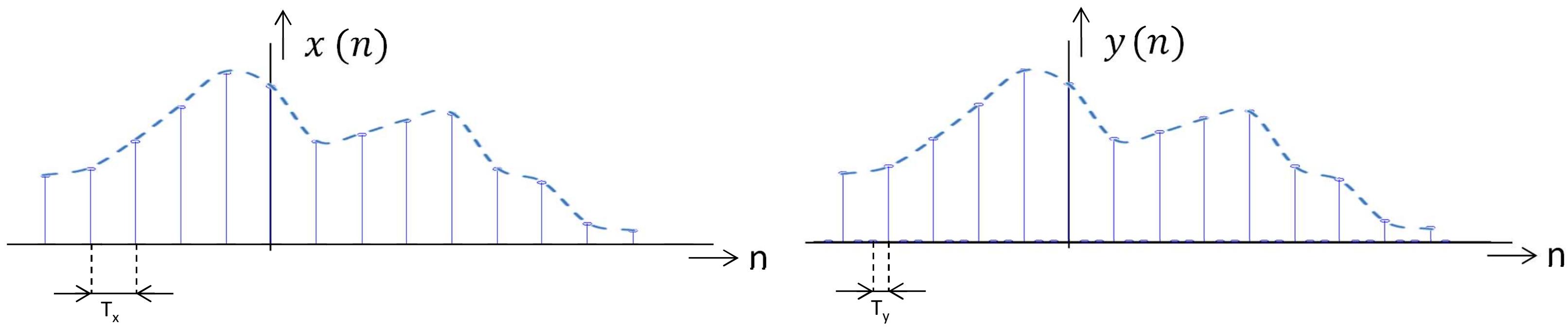


5.11

Notation



$$y(n) = \begin{cases} x\left(\frac{n}{N}\right) & n = 0, \pm N, \pm 2N, \ldots \\ 0 & otherwise \end{cases}$$



	x(n)	y(n)
Sampling period	T_x	T_y
Normalized frequency	$\theta_x = \omega T_x$	$\theta_y = \omega T_y$
Fundamental interval	$\lvert\theta_x\rvert \le \pi$ $\lvert\omega\rvert \le \pi/T_x$	$\lvert\theta_y\rvert \le \pi$ $\lvert\omega\rvert \le \pi/T_y$

Relations:

$$T_y = \frac{1}{N} T_x$$

$$\theta_y = \omega T_y = \frac{1}{N} \omega T_x = \frac{1}{N} \theta_x$$

11

Spectral representation of the SRI

5.12

$$Y(\theta_y) = \sum_{n=-\infty}^{\infty} y(n)e^{-jn\theta_y} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{N}\right) e^{-jn\theta_y}$$

$$= \sum_{k=-\infty}^{\infty} x(k)e^{-jNk\theta_y}$$

$$X(\theta_x) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta_x}$$

$$= \sum_{k=-\infty}^{\infty} x(k)e^{-jk\theta_x}$$

$$X(N\theta_y) = \sum_{k=-\infty}^{\infty} x(k)e^{-jkN\theta_y}$$

$$\rightarrow Y(\theta_y) = X(N\theta_y)$$

Therefore: $Y(\theta_y) = X(N\theta_y)$

Or: $Y(\theta_y) = X(\theta_x)$

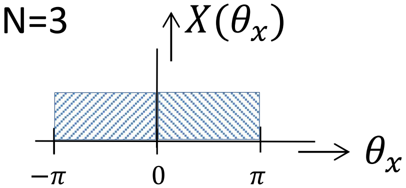
Or: $Y(\omega T_y) = X(\omega T_x)$

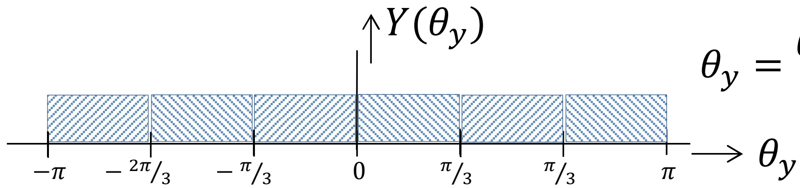
Output spectrum = input spectrum

Fundamental interval of output spectrum = N periods of fundamental interval of input

5.13

N=3



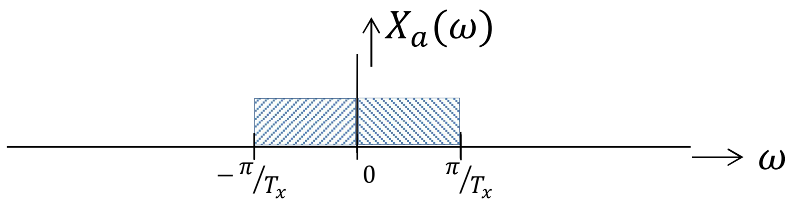


$\theta_y = {}^{\theta_x}/_3$

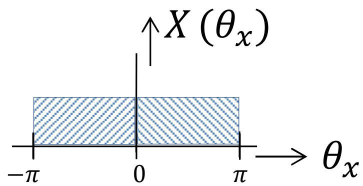
How can we find an interpolated version of x(n)?

Assume:

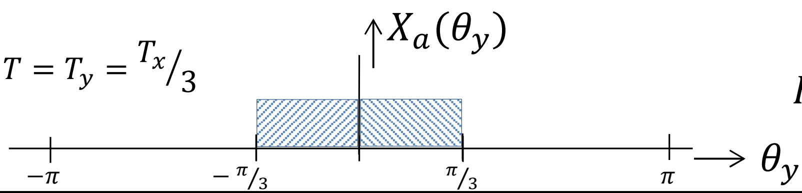
- x(n) is a sampled version of $x_a(t)$
- $x_a(\omega)$ is band limited; $x_a(\omega)=0$ for $|\omega| \geq {}^{\pi}/_{T_x}$



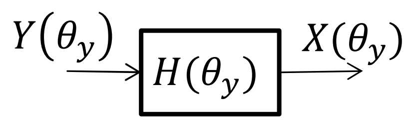
$T = T_x$



$T = T_y = {}^{T_x}/_3$



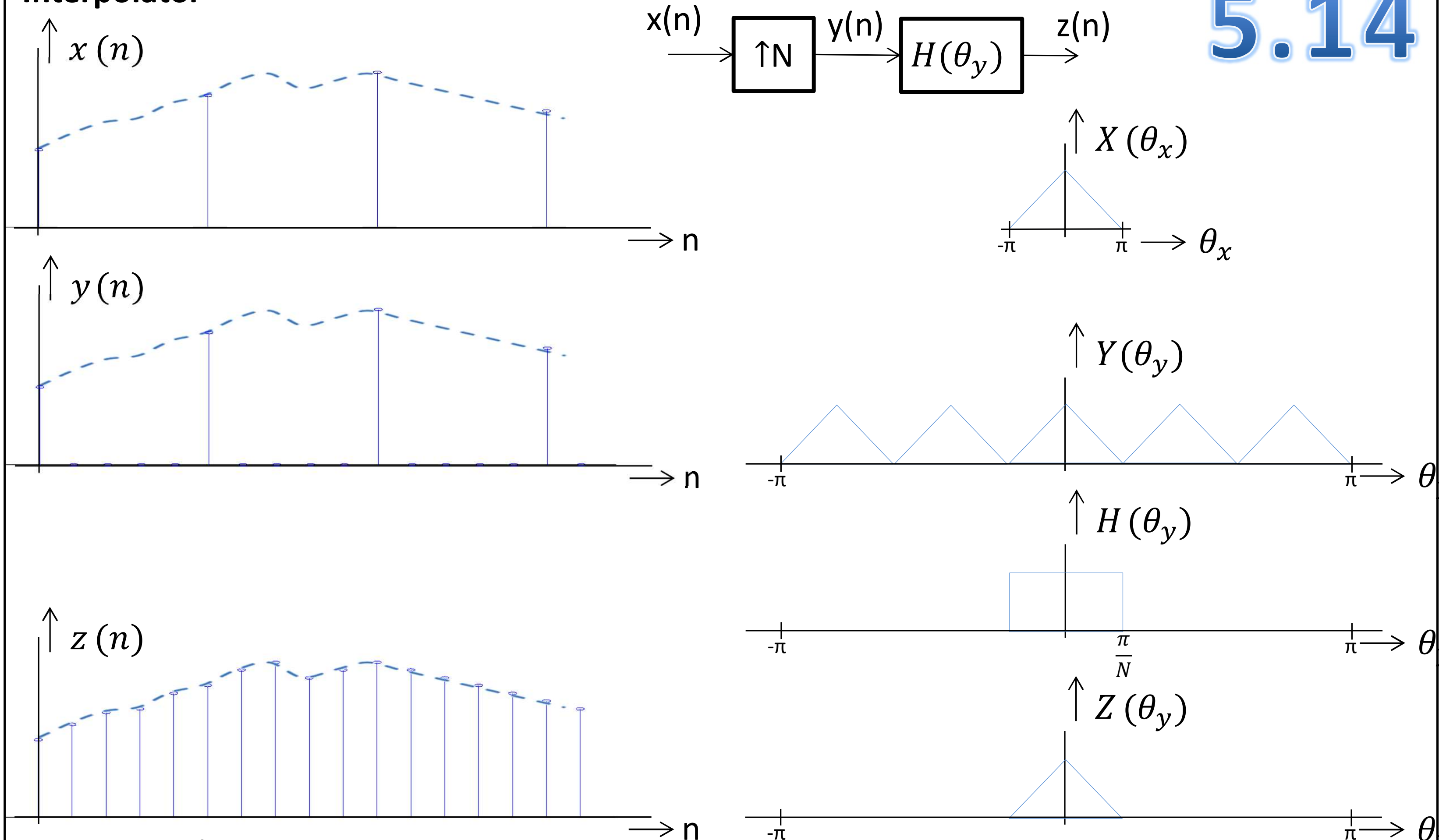
We can find $X(\theta_y)$ from $Y(\theta_y)$:



$$H(\theta_y) = \begin{cases} 1 & for \quad |\theta_y| \leq {}^{\pi}/_3 \\ 0 & for \quad {}^{\pi}/_3 \leq |\theta_y| \leq \pi \end{cases}$$

Interpolator

5.14



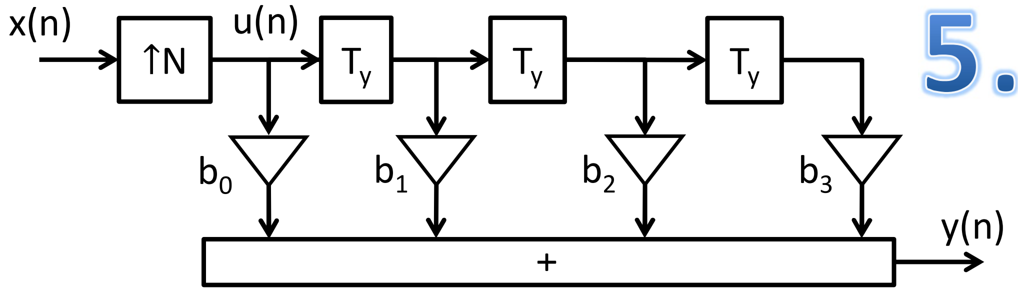
In general:

$$H(\theta_y) = \begin{cases} 1 & for \quad |\theta_y| \le \pi/N \\ 0 & for \quad \pi/N \le |\theta_y| \le \pi \end{cases}$$

Remark: The ideal low-pass filter does not exist.
In practice: SRI is cascade with an actual low pass filter = interpolation filter.

Implementation aspects

5.15

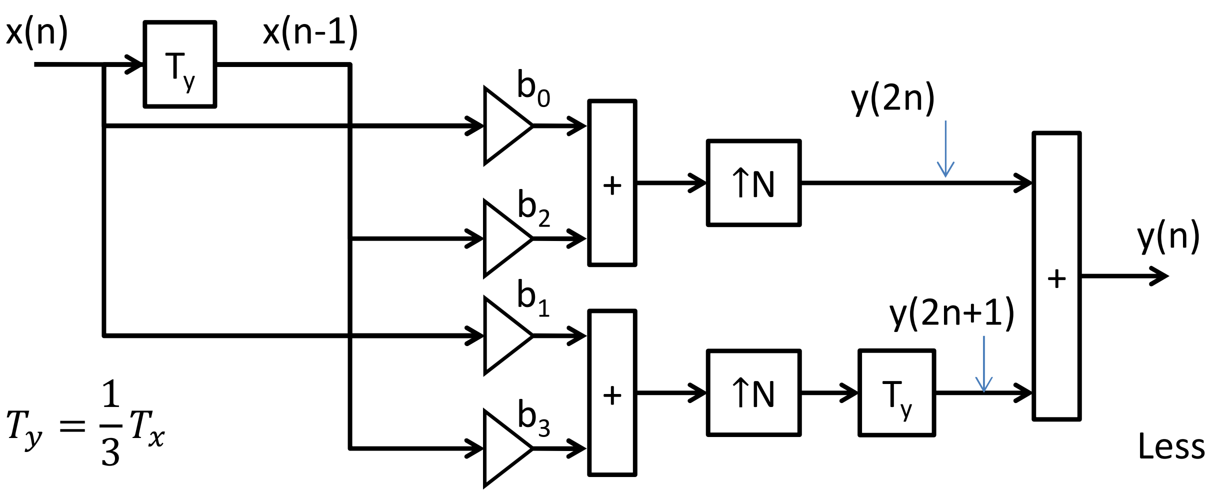


$$y(0) = b_0 u(0) + b_1 u(-1) + b_2 u(-2) + b_3 u(-3) = b_0 x(0) + b_2 x(-1)$$
$$y(1) = b_0 u(1) + b_1 u(0) + b_2 u(-1) + b_3 u(-2) = b_1 x(0) + b_3 x(-1)$$
$$y(2) = b_0 x(1) + b_2 x(0)$$
$$y(3) = b_1 x(1) + b_3 x(0)$$
$$\vdots$$

$$y(2n) = b_0 x(n) + b_2 x(n-1)$$
$$y(2n+1) = b_1 x(n) + b_3 x(n-1)$$

or

$$y(n') = b_0 x\left(\frac{n'}{2}\right) + b_2 x\left(\frac{n'}{2} - 1\right) \qquad n' = even$$
$$y(n') = b_1 x\left(\frac{n'-1}{2}\right) + b_3 x\left(\frac{n'-3}{2}\right) \qquad n' = odd$$



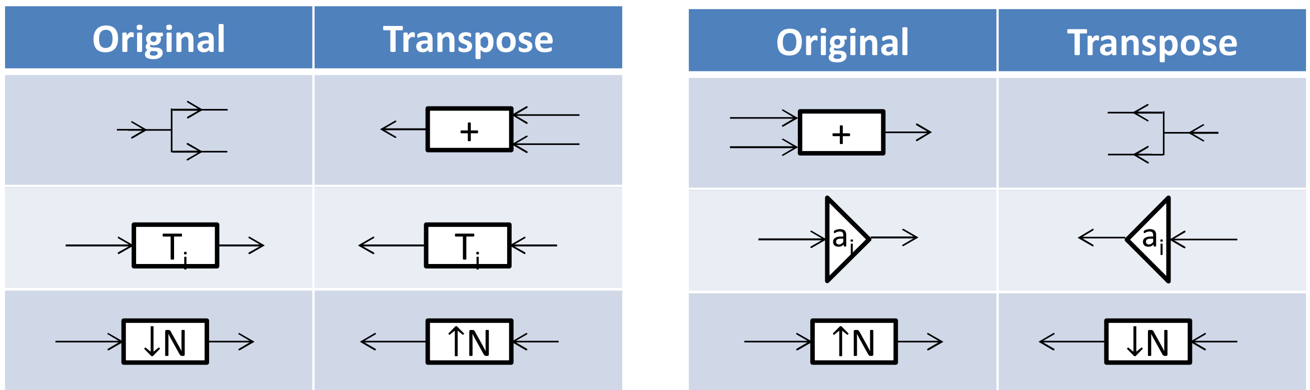
$$T_y = \frac{1}{3} T_x$$

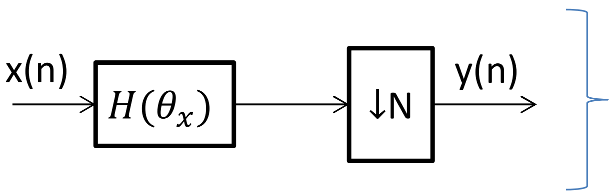
Less multiplications (50%)

Implementation aspects for recursive interpolating filters

5.16

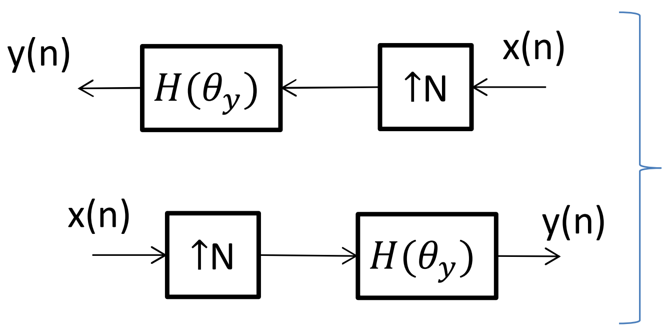
Extended transposition theorem





Decimating filter $T_y = NT_x$

⇓ Transposition

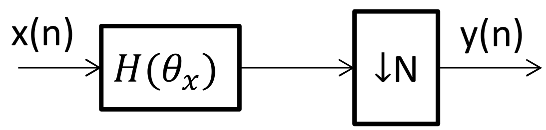


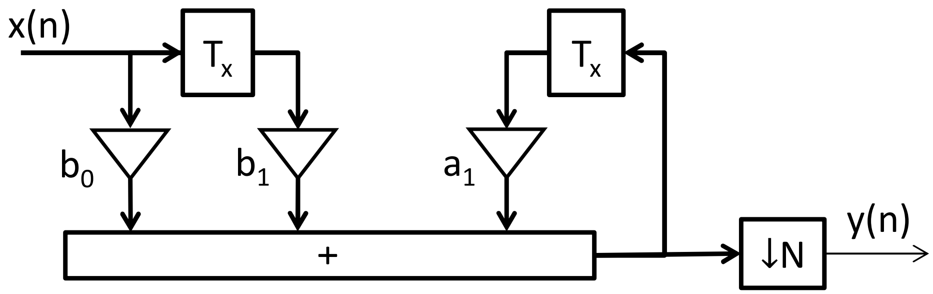
Interpolating filter $T_y = \frac{1}{N} T_x$

Transposition

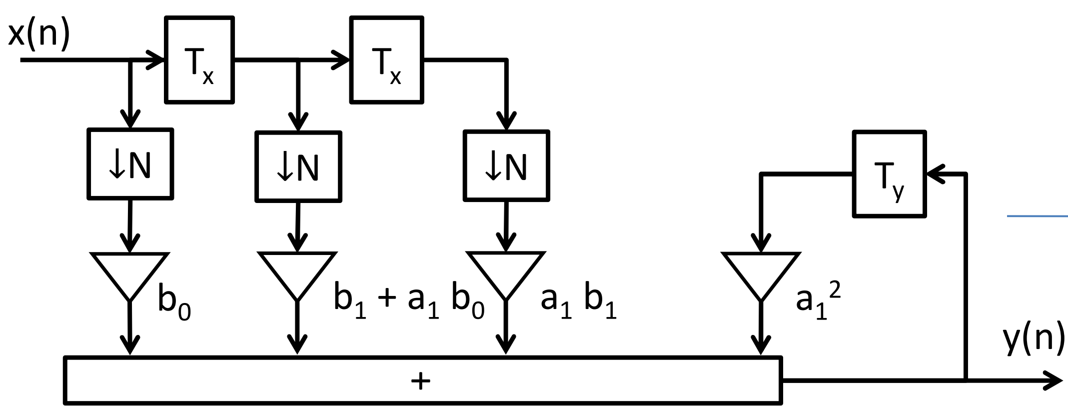
5.17

Decimating recursive filter



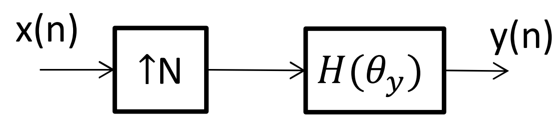


⇓ from slide 5.9

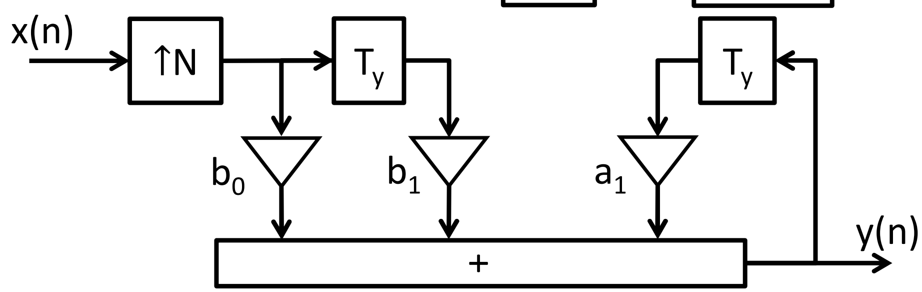


5.18

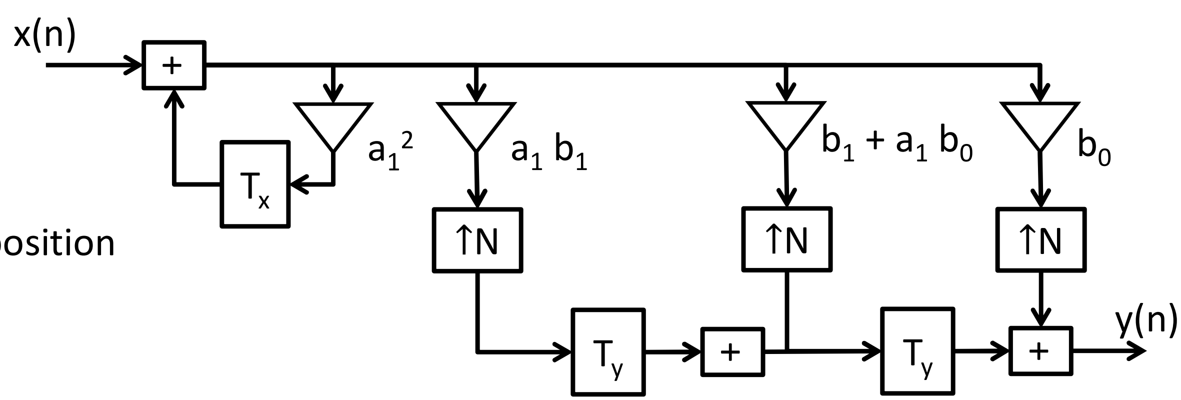
Transposition: Interpolating recursive filter



Transposition



⇓



Transposition

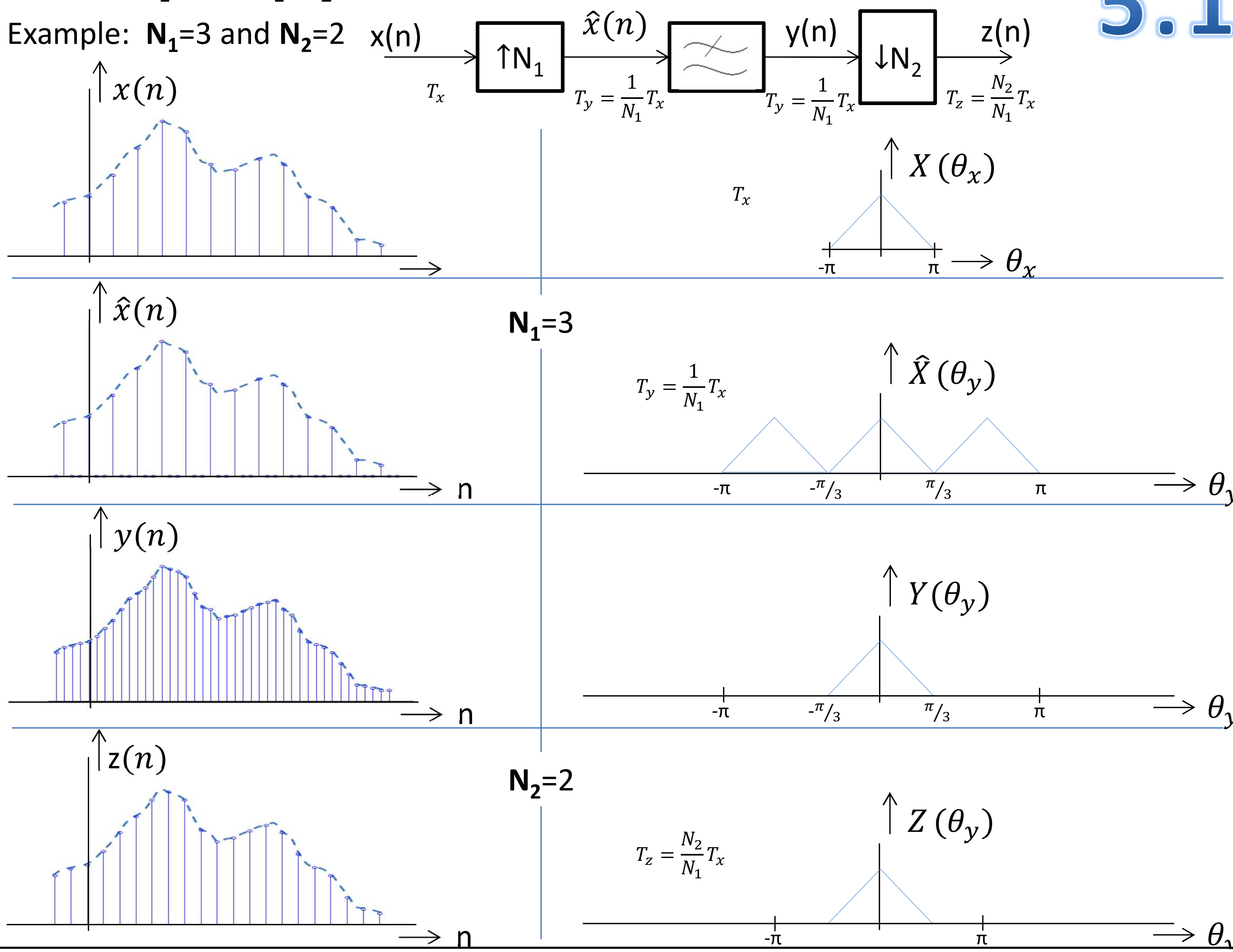
Less multiplications (4 instead of 6 per T_y seconds)

Change of F_1 by $k=N_1/N_2$

Example: N_1=3 and N_2=2

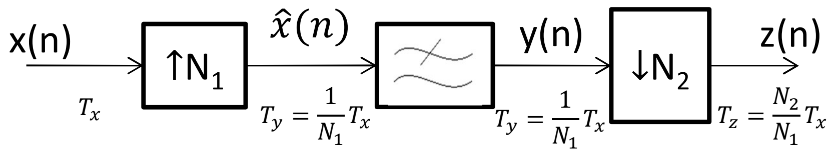
SRI + low-pass filter + SRD

5.19



19

5.20



Two cases:

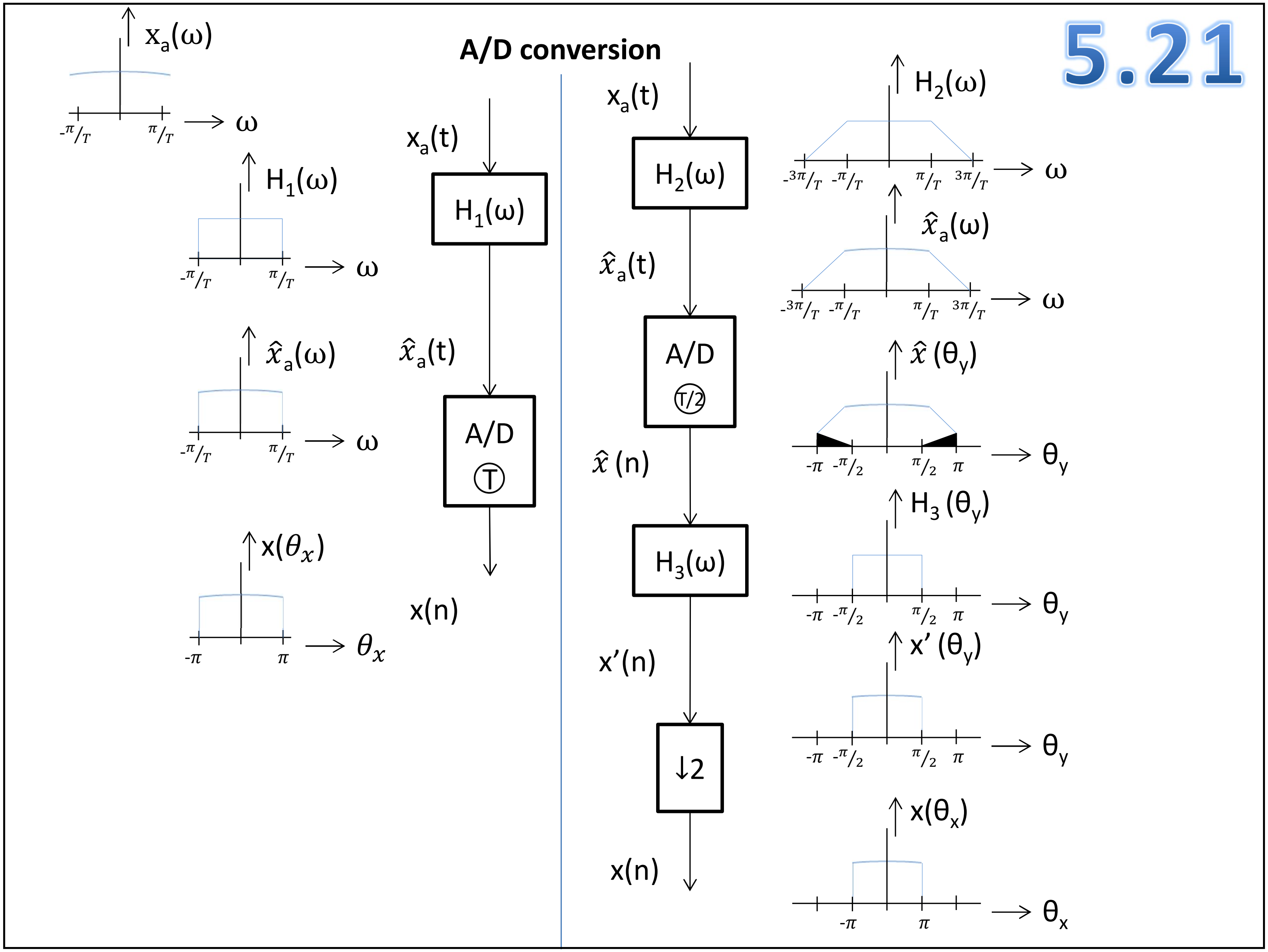
$N_1 > N_2$ (increase)

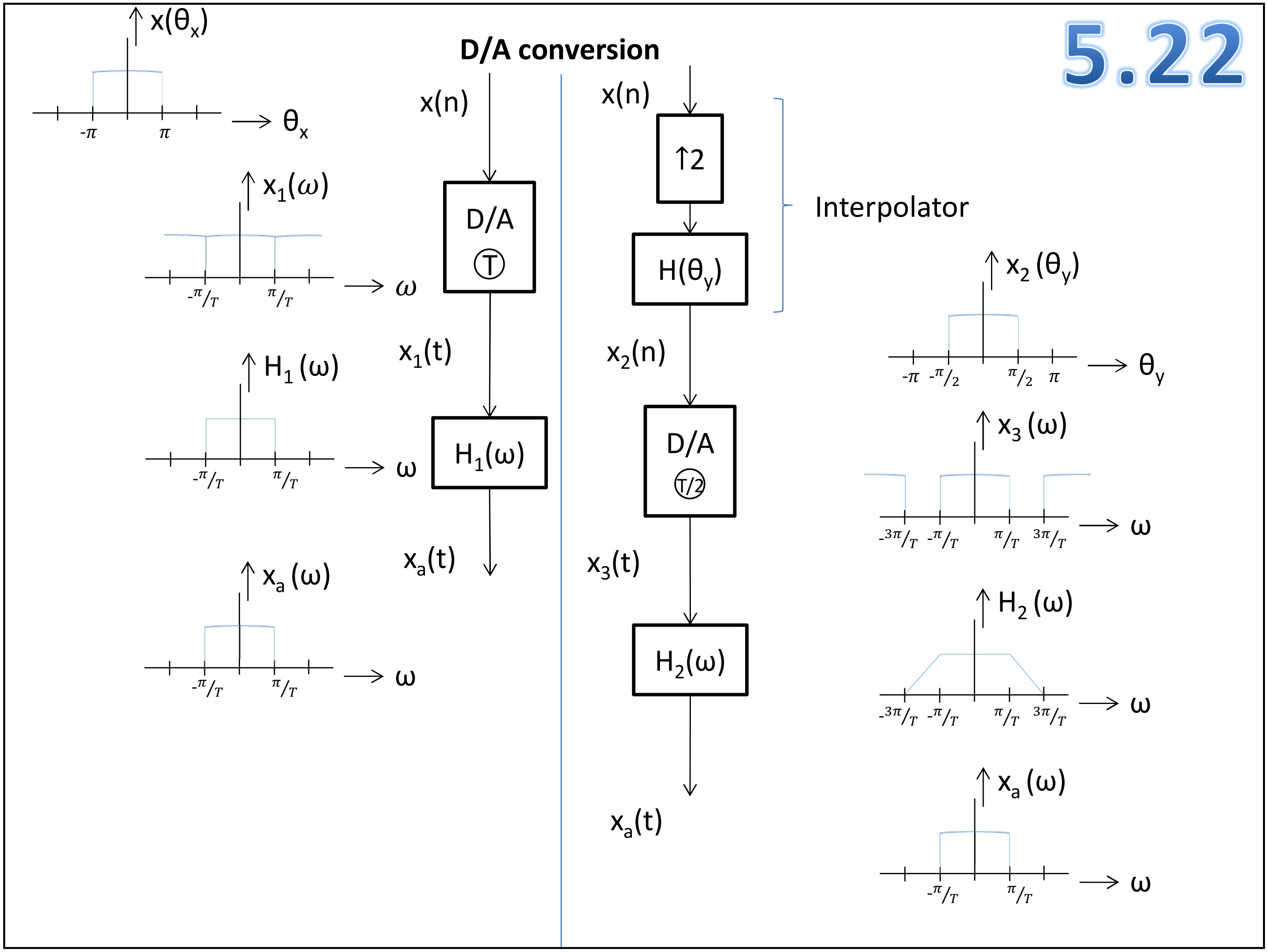
$$Z(\omega T_z) = \begin{cases} \frac{1}{N_2} X(\omega T_x) & 0 \le |\omega| \le \pi/T_x \\ 0 & \pi/T_x \le |\omega| \le \pi/T_z \end{cases}$$

$N_1 < N_2$ (decrease)

- In general: aliasing
- If spectrum of *x(n)* is band-limited to $|\omega| \le \pi/T_z$ then no aliasing:

$$Z(\omega T_z) = \frac{1}{N_2} X(\omega T_x) \quad 0 \le |\omega| \le \pi/T_z$$


A/D conversion
5.21
xa(ω)
-π/T
π/T
ω
H1(ω)
x̂a(ω)
x(θx)
-π
π
θx
xa(t)
H1(ω)
x̂a(t)
A/D
T
x(n)
xa(t)
H2(ω)
x̂a(t)
A/D
T/2
x̂ (n)
H3(ω)
x'(n)
↓2
x(n)
H2(ω)
-3π/T
-π/T
π/T
3π/T
x̂a(ω)
x̂ (θy)
-π/2
π/2
θy
H3 (θy)
x' (θy)
x(θx)
θx


5.22
D/A conversion
x(θx)
θx
-π
π
x1(ω)
ω
-π/T
π/T
H1 (ω)
xa (ω)
x(n)
D/A
T
x1(t)
H1(ω)
xa(t)
↑2
H(θy)
Interpolator
x2(n)
D/A
T/2
x3(t)
H2(ω)
x2 (θy)
θy
-π/2
π/2
x3 (ω)
-3π/T
3π/T
H2 (ω)