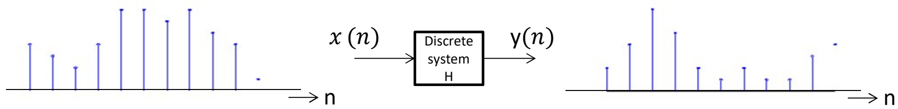
Linear time-invariant discrete (LTD) systems

3.1



Realization: Integrated circuits / Software — Microprocessor / DSP

Notation:
$$x(n) \xrightarrow{H} y(n)$$
$$x_1(n) \xrightarrow{H} y_1(n)$$
$$x_2(n) \xrightarrow{H} y_2(n)$$

H is linear if:
$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \xrightarrow{H} \alpha_1 y_1(n) \alpha_2 y_2(n)$$

From $x_1(n) = x_2(n) = 0$

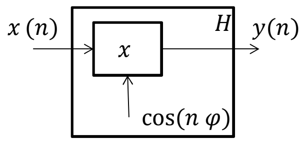
It follows
$$x(n) = 0 \xrightarrow{H} y(n) = 0$$

H is time-invariant if:
$$x(n-n_0) \xrightarrow{H} y(n-n_0)$$
?

1

Examples

3.2

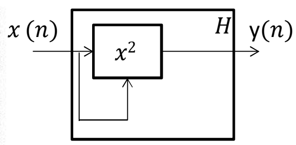


$$x_1(n) \xrightarrow{H} y_1(n) = x_1(n)\cos(n\varphi)$$
$$\varphi \neq k \cdot 2\pi$$
$$x_2(n) \xrightarrow{H} y_2(n) = x_2(n)\cos(n\varphi)$$

Linear ?
$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \xrightarrow{H} [\alpha_1 x_1(n) + \alpha_2 x_2(n)]\cos(n\varphi) = \alpha_1 x_1(n)\cos(n\varphi) + \alpha_2 x_2(n)\cos(n\varphi)$$
$$= \alpha_1 y_1(n) + \alpha_2 y_2(n)$$ Yes

Time-invariant ?
$$x(n-n_0) \xrightarrow{H} x(n-n_0)\cos(n\varphi) \neq y(n-n_0)$$

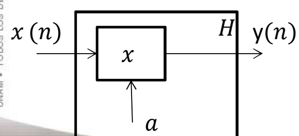
No, because: $y(n-n_0) = x(n-n_0)\cos(n-n_0)\varphi$



$$y(n) = x(n)^2$$

Linear ?
$$\alpha\, x(n) \xrightarrow{H} \alpha^2 x(n)^2 = \alpha^2 y(n) \neq \alpha\, y(n)$$ No

Time-invariant ?
$$x(n-n_0) \xrightarrow{H} x(n-n_0)^2 = y(n-n_0)$$ Yes



$$y(n) = ax(n)$$

Linear/Time-invariant

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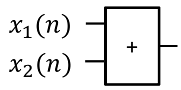
3.3

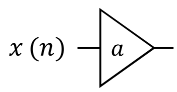
Every LTD-system can be constructed from three basic elements

Adder



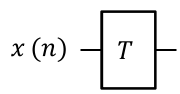
$y(n) = x_1(n) + x_2(n)$

Multiplier



$y(n) = ax(n)$

Unit delay



$y(n) = x(n-1)$

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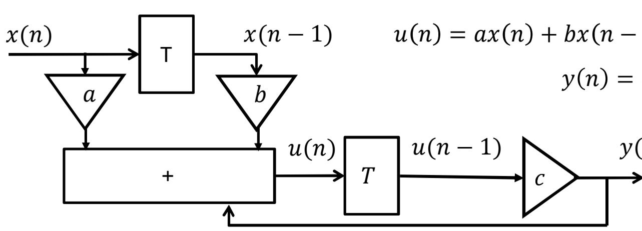
3.4

Difference equations



$u(n) = ax(n) + bx(n-1) + y(n)$ (1)

$y(n) = cu(n-1)$ (2)

From (1) and (2):

$y(n) = acx(n-1) + bcx(n-2) + cy(n-1)$

Take: $x(n) = \delta(n)$

n	$acx(n-1)$	$bcx(n-2)$	$cy(n-1)$	$y(n)$
0	0	0	0	0
1	ac	0	0	ac
2	0	bc	ac^2	$(ac+b)c$
3	0	0	$(ac+b)c^2$	$(ac+b)c^2$
4	0	0	$(ac+b)c^3$	$(ac+b)c^3$

$$y(n) = \begin{cases} 0 & n \le 0 \\ ac & n = 1 \\ (ac+b)c^{n-1} & n \ge 2 \end{cases}$$

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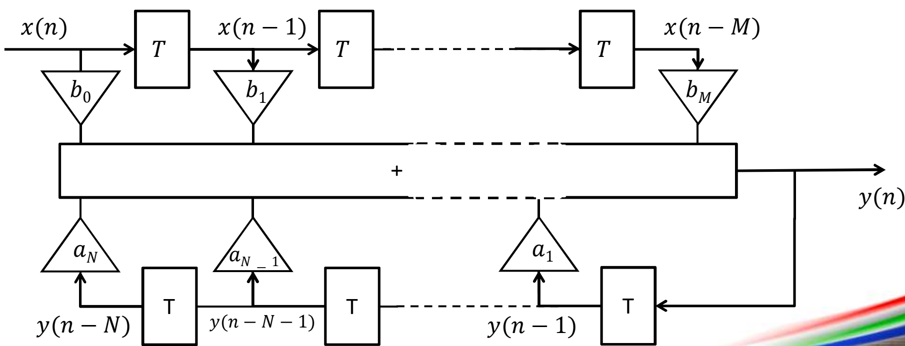
Difference equation 3.5

$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_Mx(n-M) + a_1y(n-1) + \cdots + a_Ny(n-N)$$

$$= \sum_{i=0}^{M} b_i x(n-i) + \sum_{i=0}^{N} a_i y(n-i)$$ (Linear difference equations with constant coefficients)

- Given $x(n)$; find $y(0)$, $y(1)$,… (see previous slide)
- Solve the difference equation

Remark: From the differential equation an implementation can easily be found



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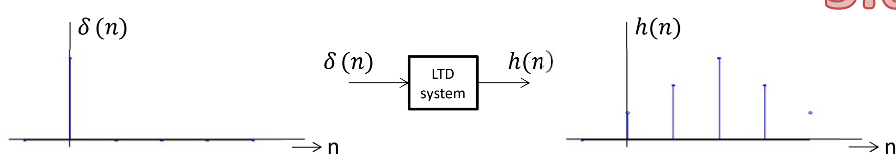
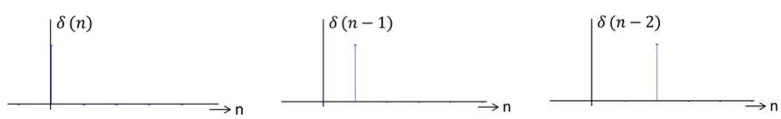
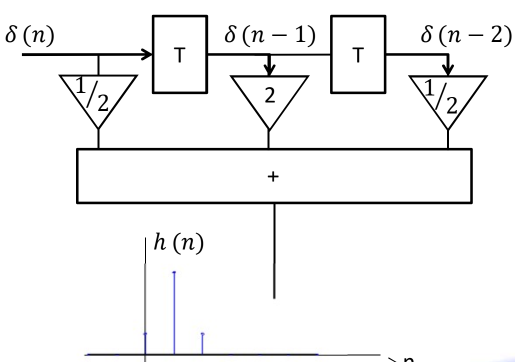
Impulse response 3.6



Example





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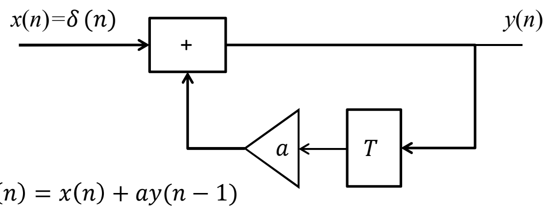
3.7

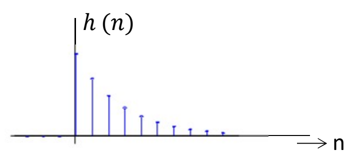
Example



$$y(n) = x(n) + ay(n-1)$$
$$y(n) = 0 \quad n < 0$$
$$x(n) = \delta(n)$$

n	x(n)	ay(n-1)	y(n)
0	1	0	1
1	0	a	a
2	0	a^2	a^2
3	0	a^3	a^3
⋮	⋮	⋮	⋮

$$y(n) = h(n) = \begin{cases} 0 & for \quad n < 0 \\ a^n & for \quad n \geq 0 \end{cases} = a^n u(n)$$



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3.8

Calculate $y(n)$ from $x(n)$ and $h(n)$

Rewrite: $x(n) = \sum_{i=-\infty}^{\infty} x(i)\delta(n-i)$

Because of linearity and time invariance:

$$\delta(n) \xrightarrow{H} h(n)$$
$$\delta(n-i) \xrightarrow{H} h(n-i)$$
$$x(i)\delta(n-i) \xrightarrow{H} x(i)h(n-i)$$
$$\underbrace{\sum_{i=-\infty}^{\infty} x(i)\delta(n-i)}_{x(n)} \xrightarrow{H} \underbrace{\sum_{i=-\infty}^{\infty} x(i)h(n-i)}_{y(n)}$$

So:

$$x(n) \xrightarrow{H} y(n) = \sum_{i=-\infty}^{\infty} x(i)h(n-i) = x(n) * h(n)$$
$$= \sum_{i=-\infty}^{\infty} h(i)x(n-i) = h(n) * x(n)$$

Discrete convolution

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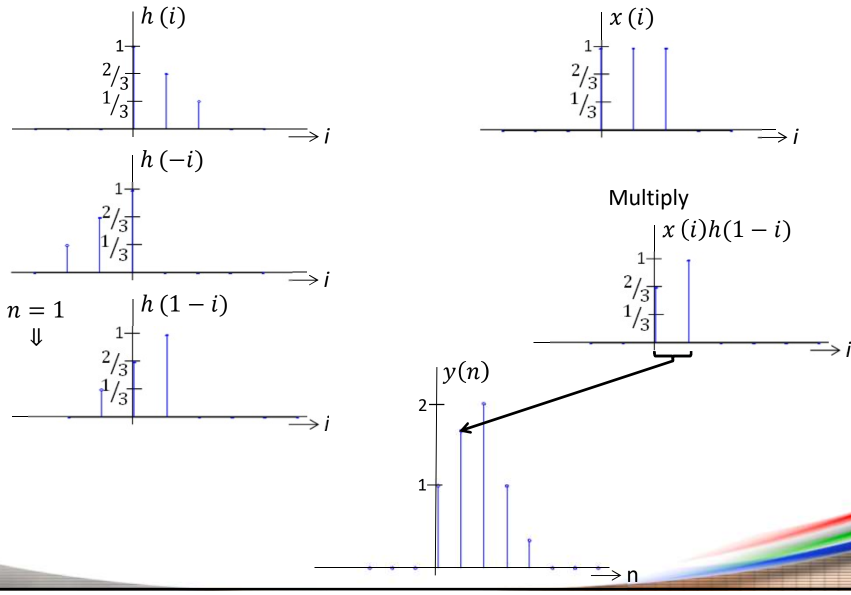
3.9

Example

$$y(n) = \sum_{i=-\infty}^{\infty} x(i)h(n-i) \qquad n = 1 \quad y(1) = \sum_{i} x(i)h(1-i)$$



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3.10

Causality

An important aspect of all physically realizable systems is causality

Definition

A system is causal if and only if the response at $n = n_0$ is not dependent on $x(n_i)$ for all $n_i > n_0$

Proposition

An LTD system is causal if and only if $h(n) = 0$ for $n < 0$

Proof

Necessary: Suppose $h(n) \neq 0$ for some $n_0 < 0$

Take $x(n) = x(0)\delta(n) \xrightarrow{H} y(n) = x(0)h(n)$

Or $y(n_0) = x(0)h(n_0) \neq 0$

The value of $y(n)$ for $n = n_0 < 0$ is determined by $x(n)$ for $n = 0$, therefore the system is not causal

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3.11

Sufficient

Let $h(n) = 0$ for $n < 0$

Then:
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} h(k)x(n-k) = h(0)x(n) + h(1)x(n-1) + \cdots$$

Therefore:

If $h(n) = 0$ for $n < 0$ it follows that $y(n)$ is determined by the present and past values of $x \rightarrow$ causal system

Causal$\leftrightarrow$ $h(n) = 0$ for $n < 0$

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3.12

Stability

A system is stable if and only if for every bounded input the output is bounded too:
$$\forall x(n);\ |x(n)| < M_x \overset{H}{\rightarrow} y(n);\ |y(n)| < M_y$$

Proposition

An LTD system is stable if and only if $\sum_{n=-\infty}^{\infty} |h(n)| < M_h$

Proof
Necessary

Assume $\sum_n |h(n)| = \infty$

Take $x(n) = sign\{h(-n)\} = \begin{cases} 1 & for \quad h(-n) \geq 0 \\ -1 & for \quad h(-n) < 0 \end{cases}$

$$y(n) = \sum_i h(i)x(x-i)$$

Or: $y(0) = \sum_i h(i)x(-i) = \sum_i |h(i)| = \infty$

Therefore: for this bounded input, the output is not bounded

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3.13

Sufficient

Assume $\sum_n |h(n)| < M_n$

$$y(n) = \sum_k h(k)x(n-k)$$

$$|y(n)| = \left|\sum_k h(k)x(n-k)\right| \leq \sum_k |h(k)x(n-k)| = \sum_k |h(k)||x(n-k)|$$

With $|x(n)| \leq M_x$:

$$|y(n)| \leq \sum_k M_x|h(k)| = M_x \sum_k |h(k)|$$

$$|y(n)| \leq M_x M_h$$

Stability $\leftrightarrow \sum_{n=-\infty}^{\infty} |h(n)| < M_h$

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3.14

Proposition
If in a stable LTD system $|x(n)| < M_x$ and $x(n) = 0$ for $n > n_0$ then:

$$\lim_{n\to\infty} y(n) = 0$$

Proof

$$y(n_0 + N) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 + N - k) = \sum_{k=N}^{\infty} h(k)x(n_0 + N - k)$$

Or:

$$|y(n_0 + N)| \leq M_x \sum_{k=N}^{\infty} |h(k)|$$

Since H is stable:

$$\sum_{k=-\infty}^{\infty} |h(k)| < M_h \qquad \lim_{N\to\infty} \sum_{k=N}^{\infty} |h(k)| = 0$$

We see:

$$\lim_{N\to\infty} |y(n_0 + N)| = M_x \lim_{N\to\infty} \sum_{k=N}^{\infty} |h(k)| = 0 \qquad \lim_{N\to\infty} y(n) = 0 \quad Q.E.D.$$

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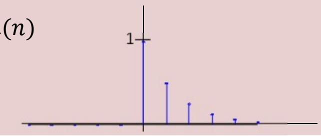
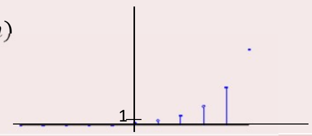
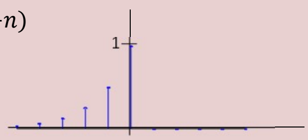
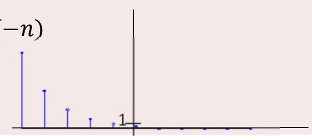
Examples

3.15

h(n)	Causal	Stable
$h(n) = 2^{-n}u(n)$ 	Yes	Yes
$h(n) = 2^{n}u(n)$ 	Yes	No
$h(n) = 2^{n}u(-n)$ 	No	Yes
$h(n) = 2^{-n}u(-n)$ 	No	No

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3.16

Transmission function H(θ) $H(\theta) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\theta}$ The FTD of $h(n)$

Response to $x(n) = \cos(n\theta)$:

$$y(n) = x(n) * h(n) = \sum_k x(n-k)h(k) = \sum_k \cos[(n-k)\theta]h(k)$$

$$= \sum_k \frac{1}{2}\left[e^{j(n-k)\theta} + e^{-j(n-k)\theta}\right]h(k) = \frac{1}{2}e^{jn\theta}\sum_k h(k)e^{-jk\theta} + \frac{1}{2}e^{-jn\theta}\sum_k h(k)e^{jk\theta}$$

$$= \frac{1}{2}e^{jn\theta}H(\theta) + \frac{1}{2}e^{-jn\theta}H(-\theta)$$

Remark: $h(n)$ is real → $H(\theta) = A(\theta)\, e^{j\Psi(\theta)}$ → $H(-\theta) = A(\theta)\, e^{-j\Psi(\theta)}$

$$y(n) = \frac{1}{2}A(\theta)\left[e^{j(n\theta+\Psi(\theta))} + e^{-j(n\theta+\Psi(\theta))}\right] = A(\theta)cos[n\theta + \Psi(\theta)]$$

$$x(n) = \cos(n\theta) \xrightarrow{H} \underline{A(\theta)}cos[n\theta + \underline{\Psi(\theta)}]$$

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3.17

For an arbitrary $x(n)$ $\qquad y(n) = x(n) * h(n)$

Or: $\quad Y(\theta) = X(\theta)H(\theta)$

Proof:

$$Y(\theta) = \sum_{n} y(n)e^{-jn\theta} = \sum_{n}\sum_{k} x(k)h(n-k)\, e^{-jn\theta} = \sum_{k}\sum_{n} x(k)h(n-k)\, e^{-j(n-k)\theta}e^{-jk\theta}$$

$$= \sum_{k} x(k)e^{-jk\theta} \sum_{n} h(n-k)\, e^{-j(n-k)\theta}$$

$$n - k = l$$

$$= \sum_{k} x(k)e^{-j} \quad \sum_{n} h(l)\, e^{-jl\theta} = X(\theta)H(\theta)$$

Alternatively: $\qquad |Y(\theta)| = |X(\theta)||H(\theta)|$

$$Arg\big(Y(\theta)\big) = Arg\big(X(\theta)\big) + Arg(H(\theta))$$

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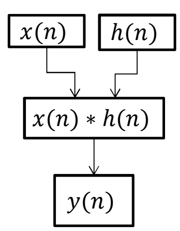
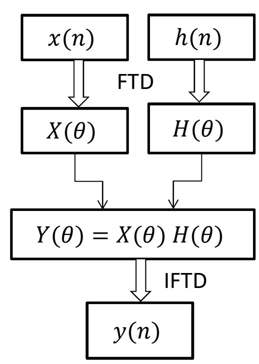
3.18

Resume:





How to determine $H(\theta)$:

1. Take the FTD of $h(n)$
2. Take $x(n) = \cos(n\,\theta)$ and $y(n) = |H|\cos(n\theta + \Psi)$ and substitute this in the difference equation
   Alternative:
   $$x(n) = e^{jn\theta} \rightarrow y(n) = H(\theta)e^{jn\theta}$$
3. Use the difference equation
   Apply time shift property
   We find an equation in $X(\theta)$ and $Y(\theta)$
   Write
   $$H(\theta) = \frac{Y(\theta)}{X(\theta)}$$

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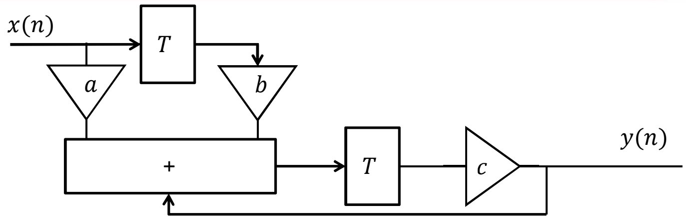
3.19

Example



$$y(n) = acx(n-1) + bcx(n-2) + cy(n-1) \qquad h(n) = \begin{cases} 0 & n \leq 0 \\ ac & n = 1 \\ (ac+b)c^{n-1} & n \geq 2 \end{cases}$$

1. $H(\theta) = \sum_{n} h(n)e^{-j} \quad = ace^{-j\theta} + \sum_{n=2}^{\infty}(ac+b)c^{n-1}e^{-jn\theta}$

$$= ace^{-j\theta} + (ac+b)e^{-j\theta}\sum_{n=2}^{\infty}\left(ce^{-j\theta}\right)^{n-1} = ace^{-j\theta} + (ac+b)e^{-j}\ \frac{ce^{-j\theta}}{1-ce^{-j}}$$

$$= \frac{ace^{-j\theta}\left(1-ce^{-j}\ \right) + (ac+b)e^{-j}\ ce^{-j\theta}}{1-ce^{-j}}$$

$$\boxed{\mathrm{H}(\theta) = \frac{ace^{-j}\ + bce^{-2j\theta}}{1-ce^{-j\theta}}}$$

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3.20

2. $x(n) = \cos(n\theta) \xrightarrow{H} y(n) = |H|\cos(n\theta + \Psi)$

(Very complicated: see survey page 3.10)

$$x(n) = e^{jn}\ \xrightarrow{H} y(n) = H(\theta)e^{jn\theta}$$

$$The\ D.E. \quad y(n) = acx(n-1) + bcx(n-2) + cy(n-1)$$

$$H(\theta)e^{jn\theta} = ace^{j(n-1)\theta} + bce^{j(n-2)\theta} + cH(\theta)e^{j(n-1)\theta}$$

$$= ace^{-j\theta}e^{jn\theta} + bce^{-2j\theta}e^{jn\theta} + cH(\theta)e^{-j\theta}e^{jn\theta}$$

$$H(\theta)\left[1-ce^{-j\theta}\right] = ace^{-j\theta} + bce^{-2j\theta}$$

$$\boxed{H(\theta) = \frac{ace^{-j\theta} + bce^{-2j\theta}}{1-ce^{-j\theta}}}$$

3. $y(n) = acx(n-1) + bcx(n-2) + cy(n-1)$

Use FTD

$$Y(\theta) = ace^{-j\theta}X(\theta) + bce^{-2j\theta}X(\theta) + ce^{-j\theta}Y(\theta)$$

$$\boxed{\frac{Y(\theta)}{X(\theta)} = H(\theta) = \frac{ace^{-j\theta} + bce^{-2j\theta}}{1-ce^{-j\theta}}}$$

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3.21

System function $\widetilde{H}(z)$

$$\widetilde{H}(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \quad [The\ ZT\ of\ h(n)]$$

The transmission function $H(\theta)$ = the system function $\widetilde{H}(z)$ evaluated on the unit circle of the z-plane:

$$\widetilde{H}(z)\Big|_{z\,=\,e^{j\theta}} = \widetilde{H}(e^{j\theta}) = H(\theta)$$

Response to an arbitrary $x(n)$: $\quad y(n) = x(n) * h(n) \xrightarrow{H} \tilde{Y}(z) = \tilde{X}(z)\,\widetilde{H}(z)$

Proof:

$$\tilde{Y}(z) = \sum_n y(n) z^{-n} = \sum_n \sum_k x(k) h(n-k) z^{-n} = \sum_k x(k) \sum_n h(n-k) z^{-(n-k)} z^{-k}$$

$$n - k = l$$

$$= \sum_k x(k)\, z^{-k} \sum_n h(l) z^{-l} = \tilde{X}(z)\,\widetilde{H}(z)$$

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3.22

How to determine H(z):

1. The ZT of h(n)
2. $x(n) = z^n \xrightarrow{H} \sum_i h(i) x(n-i) = \sum_i h(i) z^{n-i} = z^n \sum_i h(i) z^{-i} = z^n \widetilde{H}(z)$

   Find the difference equation of the system
   Take $x(n) = z^n$ and $y(n) = z^n \widetilde{H}(z)$
   We find an expression in $\widetilde{H}(z)$

3. Use the difference equation
   Use the time shift property of the ZT

$$x(n) \Rightarrow \tilde{X}(z)$$
$$x(n-i) \Rightarrow z^{-i} \tilde{X}(z)$$

   We find an expression in $\tilde{X}(z)$ and $\tilde{Y}(z)$
   Solve this equation:

$$\frac{\tilde{Y}(z)}{\tilde{X}(z)} = \widetilde{H}(z)$$

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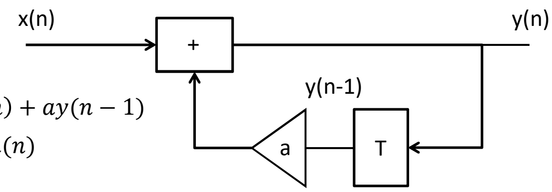
3.23

Example



$y(n) = x(n) + ay(n-1)$

$h(n) = a^n u(n)$

1. $\tilde{H}(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a\,z^{-1})^n = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$

2. $x(n) = z^n \quad en \quad y(n) = z^n \tilde{H}(z) \rightarrow y(n-1) = z^{n-1}\tilde{H}(z)$

$$y(n) = x(n) + ay(n-1)$$

$$z^n \tilde{H}(z) = z^n + az^{n-1}\tilde{H}(z)$$

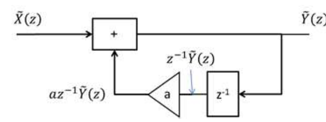
This gives:

$$\tilde{H}(z) = \frac{z^n}{z^n - az^{n-1}} = \frac{1}{1 - az^{-1}}$$

3. $y(n) = x(n) + ay(n-1)$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$\tilde{Y}(z) = \tilde{X}(z) + az^{-1}\tilde{Y}(z) \rightarrow \tilde{H}(z) = \frac{\tilde{Y}(z)}{\tilde{X}(z)} = \frac{1}{1 - az^{-1}}$$



$$\Downarrow$$

$$\tilde{Y}(z) = \tilde{X}(z) + az^{-1}\tilde{Y}(z)$$

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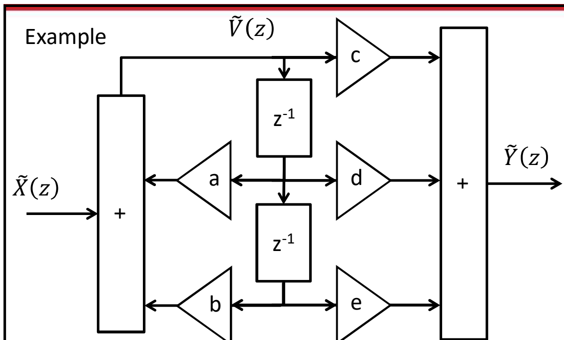
3.24

Example



Define: $\tilde{V}(z)$

We see: $\tilde{V}(z) = \tilde{X}(z) + az^{-1}\tilde{V}(z) + \mathrm{b}z^{-2}\tilde{V}(z)$

Or: $\tilde{V}(z) = \frac{\tilde{X}(z)}{1 - az^{-1} - \mathrm{b}z^{-2}}$

And: $\tilde{Y}(z) = c\tilde{V}(z) + \mathrm{d}z^{-1}\tilde{V}(z) + \mathrm{e}z^{-2}\tilde{V}(z) = (c + \mathrm{d}z^{-1} + \mathrm{e}z^{-2})\tilde{V}(z)$

$$= \frac{(c + \mathrm{d}z^{-1} + \mathrm{e}z^{-2})\tilde{X}(z)}{1 - az^{-1} - \mathrm{b}z^{-2}}$$

$$\frac{\tilde{Y}(z)}{\tilde{X}(z)} = \tilde{H}(z) = \frac{c + \mathrm{d}z^{-1} + \mathrm{e}z^{-2}}{1 - az^{-1} - \mathrm{b}z^{-2}}$$

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$\widetilde{H}(z)$ is a rational function in z:

$$\widetilde{H}(z)=\frac{a_0+a_1z+a_2z^2+\cdots+a_Mz^M}{b_0+b_1z+b_2z^2+\cdots+b_Nz^N}=\frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}\frac{a_M}{b_N}$$

Partial fraction expansion (N>M)

$$\widetilde{H}(z)=\frac{A_1}{1-p_1z^{-1}}+\frac{A_2}{1-p_2z^{-1}}+\cdots=\sum_{k=1}^{N}\frac{A_k}{1-p_kz^{-1}}$$

Or: $h(n)=\sum_{k=1}^{N}A_kP_k^n\ \ u(n)$

Stability:

$$\sum_{n=-\infty}^{\infty}|h(n)|=\sum_{n=0}^{\infty}\left|\sum_{k=1}^{N}A_kp_k^n\right|\le\sum_{n=0}^{\infty}\sum_{k=1}^{N}\left|A_kp_k^n\right|\le\sum_{n=0}^{\infty}\sum_{k=1}^{N}|A_k||p_k^n|$$

$$\le\sum_{k=1}^{N}|A_k|\sum_{n=0}^{\infty}|p_k|^n$$ Finite if $|p_k|<1$

If one pole outside $|z|=1;\ |p_k|>1$

Then $\frac{A_k}{1-p_kz^{-1}}\to(p_k)^nA_ku(n)\quad infinite\ for\quad n\to\infty$

Conclusion:

Stability ↔All poles inside $|z|=1$

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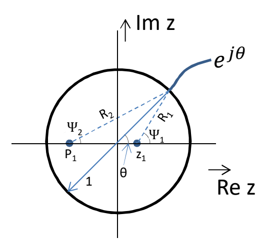
Properties of poles and zeros



1. Stability; Poles: inside $|z|=1$
   Zeroes: No restriction
2. $\widetilde{H}(z)=\frac{z-z_1}{z-p_1}\quad \widetilde{H}(e^{j\theta})=\mathrm{H}(\theta)=\frac{e^{j\theta}-z_1}{e^{j\theta}-p_1}$

   $|\mathrm{H}(\theta)|=\frac{R_1}{R_2}\quad Arg\mathrm{H}(\theta)=\Psi_1-\Psi_2$
3. Pole on unit circle: $\mathrm{H}(\theta)=\infty$
   Zero on unit circle: $\mathrm{H}(\theta)=0$ } $\triangle\{Arg\mathrm{H}(\theta)\}=\pi$
4. All zeroes inside $|z|=1$: Minimum phase network

   If the term $1-az^{-1}$ (zero for z=a) is changed into $1-\frac{z^{-1}}{a}$ (zero for z=1/a), then:

   The shape of $|\mathrm{H}(\theta)|$ is not changes, only $Arg\mathrm{H}(\theta)$ is changes

   For a minimum phase network: The group delay $-d\Psi/d\theta$ is minimal for all θ

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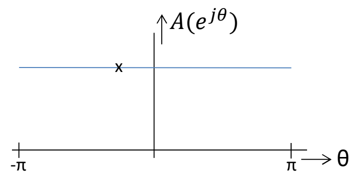
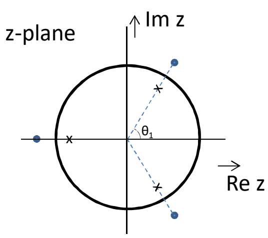
5. Stable all-pass network:

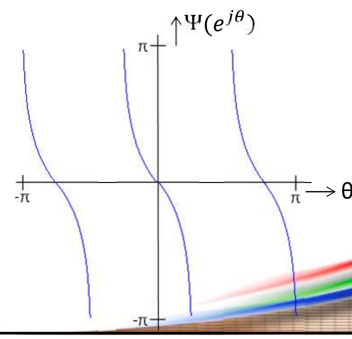
This means $|H(\theta)| = k \quad for \quad -\pi \le \theta \le \pi$

$$H(z) = \frac{\left(a - z^{-1}/a\right)\left(a - z^{-1}/b\right)\ldots}{(a - az^{-1})(a - bz^{-1})\ldots}$$

- All poles inside $|z| = 1$
- Pole-zero pair: pole z=a
  zero z=1/a
- All zeroes outside $|z| = 1$







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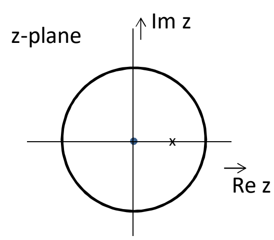
Examples

$$h(n) = a^n u(n)$$

$$\widetilde{H}(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

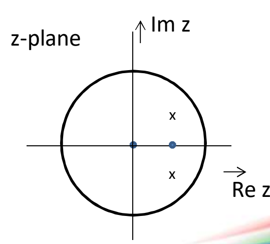
$$zero: \quad z = 0$$
$$pole: \quad z = a$$



$$h(n) = \rho^n \cos(n\varphi)\, u(n)$$

$$\widetilde{H}(z) = \frac{z(z - \rho\cos(\varphi))}{z^2 - 2\rho z cos(\varphi) + \rho^2} = \frac{z(z - \rho\cos(\varphi))}{(z - \rho cos(\varphi) - j\rho\sin(\varphi))(z - \rho cos(\varphi) + j\rho\sin(\varphi))}$$

$$zero: \quad z = 0 \quad z = \rho\cos(\varphi)$$
$$pole: \quad z = \rho cos(\varphi) \mp j\rho\sin(\varphi))$$



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