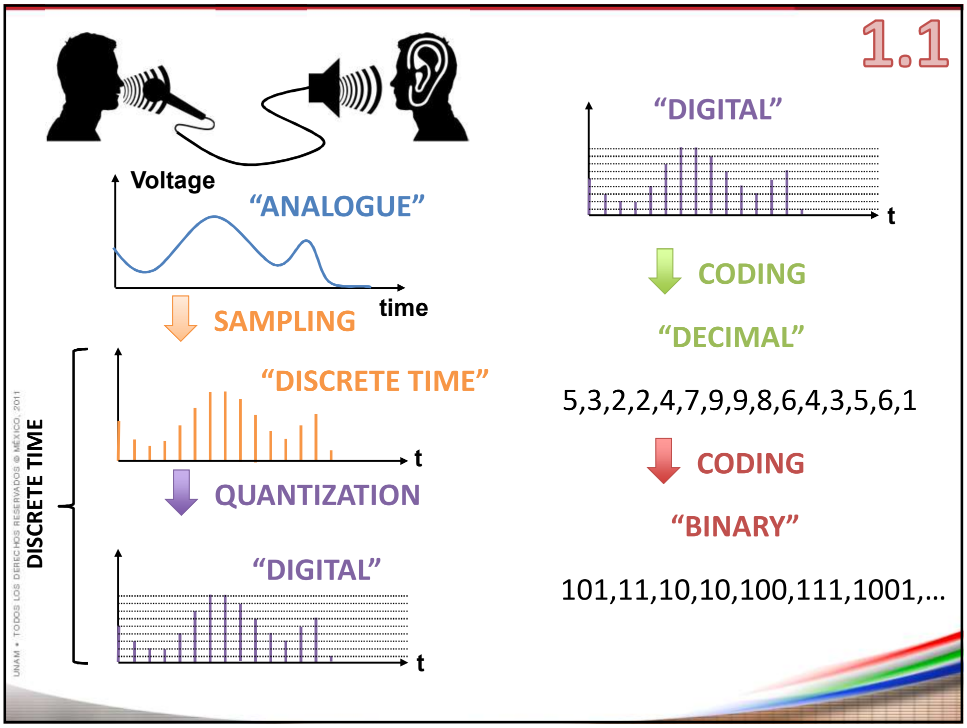
1.1
Voltage
"ANALOGUE"
time
SAMPLING
"DISCRETE TIME"
t
QUANTIZATION
"DIGITAL"
t
DISCRETE TIME
"DIGITAL"
t
CODING
"DECIMAL"
5,3,2,2,4,7,9,9,8,6,4,3,5,6,1
CODING
"BINARY"
101,11,10,10,100,111,1001,…
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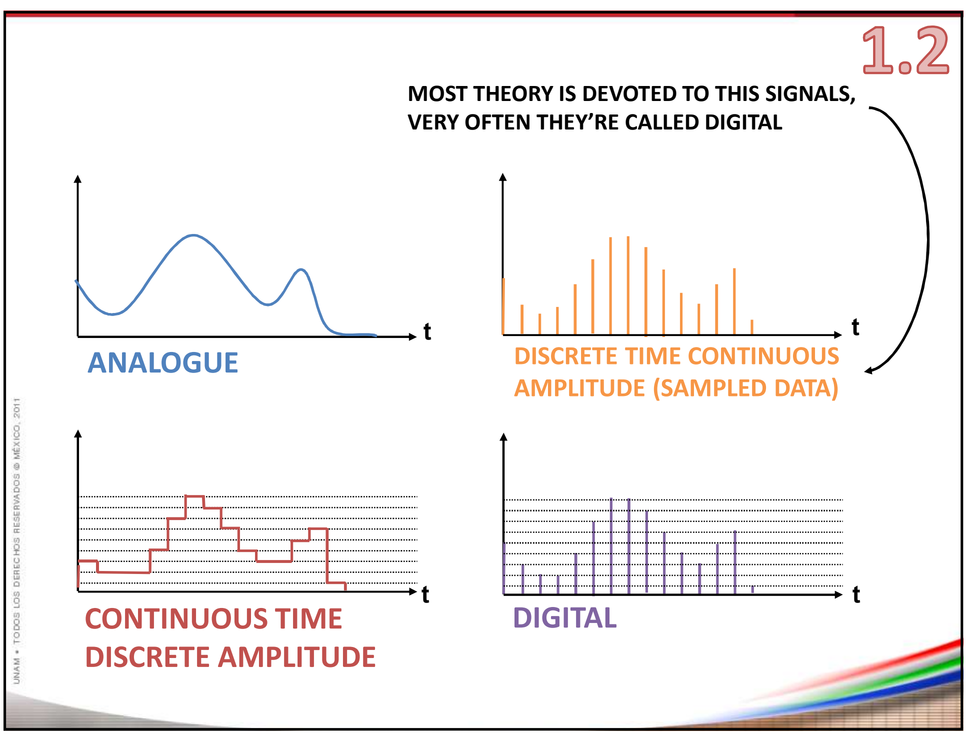
1


1.2
MOST THEORY IS DEVOTED TO THIS SIGNALS,
VERY OFTEN THEY'RE CALLED DIGITAL
t
ANALOGUE
t
DISCRETE TIME CONTINUOUS
AMPLITUDE (SAMPLED DATA)
t
CONTINUOUS TIME
DISCRETE AMPLITUDE
t
DIGITAL
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2

DIGITAL SIGNAL PROCESSING

1.3

ADVANTAGES

- IC REALIZATION (SMALL, CHEAP, RELIABLE, COMPLEX SYSTEMS POSSIBLE)
- ACCURACY (AGING, TEMPERATURE)
- PROGRAMABILITY
- MULTIPLEXING

DISADVANTAGES

- A/D AND D/A CONVERSION
- COMPLEX CIRCUITS
- POWER DISSIPATION

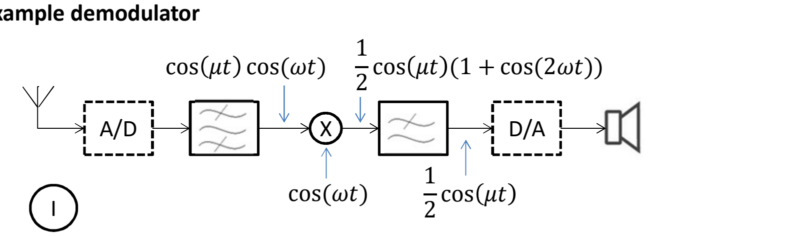
REMARKS

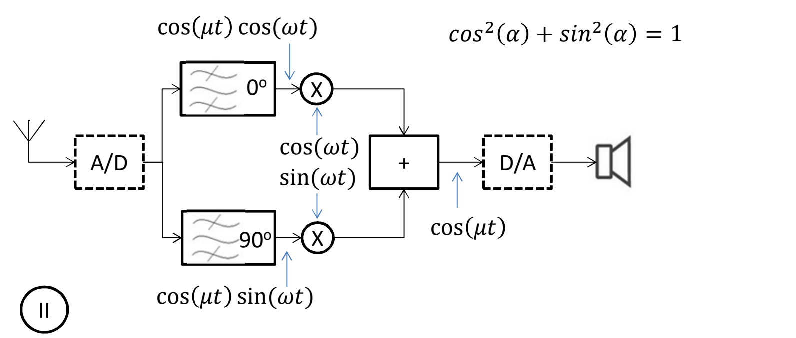
- SOFTWARE/HARDWARE REALIZATION
- REALTIME/OFF-LINE SYSTEMS
- POSSIBILITY OF NEW FUNCTIONS
  - IDEAL INTEGRATOR
  - PERFECT COMPENSATION
- ANALOG -> DIGITAL SYSTEM CONVERSION
  - EACH FUNCTION
  - WHOLE SYSTEM

3

Example demodulator

1.4





Solution II is very difficult for analog systems (Accuracy)

4

APPLICATION AREAS

1.5

- Space technology
- Seismology (finding oil)
- Radar / Sonar
- Medical Systems (EEG,ECG)
- Data transmission (modems, equalization)
- Telephony (PCM enconding, vocoders)
- Audio (voicer recognition, artificial speech)
- Video (image restoration, image recognition)
- Toys (speaking dolls, remote controlled)

5

1.6

DIGITAL SIGNALS

1. CONTINUOUS SIGNAL $x(t)$

   ⇓

   SAMPLING WITH PERIOD T

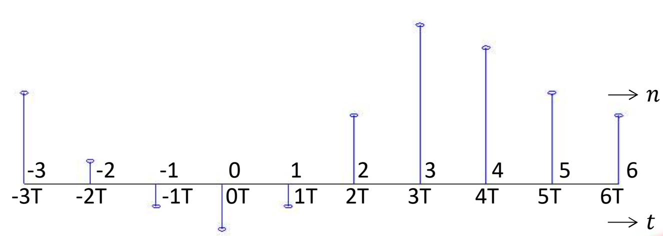
   ⇓

   DIGITAL SIGNAL $x(nT)$

2. DIGITAL SIGNAL GENERATORS

NOTATION

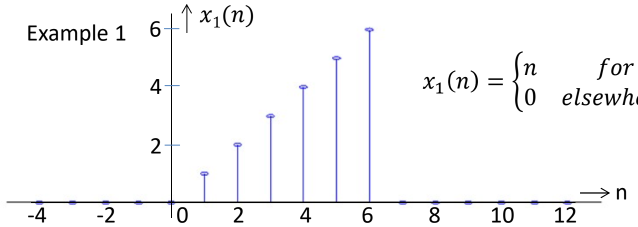
- $x(nT)$: A SEQUENCE OF SAMPLES OCCURING AT REGULAR PERIODS OF TIME
- CONVENIENT: $x(n)$
- A DIGITAL SIGNAL $x(n)$ IS:
  - DEFINED FOR ALL n
  - NOT DEFINED BETWEEN THE SAMPLES



6

Representation in the time domain

Example 1

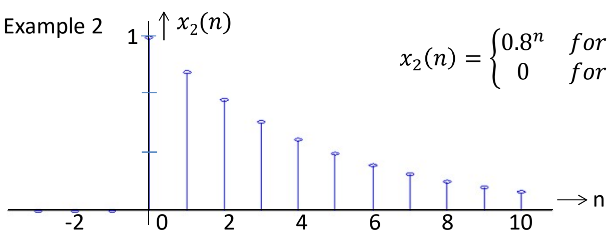


$$x_1(n) = \begin{cases} n & for \quad 0 \le n \le 6 \\ 0 & elsewhere \end{cases}$$

Finite duration

Finite duration if: $x(n) = 0 \begin{cases} n < N_1 \\ n > N_2 \end{cases}$

Example 2



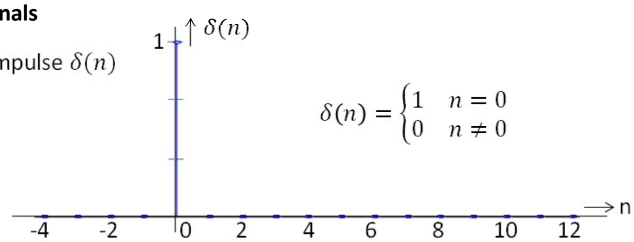
$$x_2(n) = \begin{cases} 0.8^n & for \quad n \ge 0 \\ 0 & for \quad n < 0 \end{cases}$$

Infinite duration

7

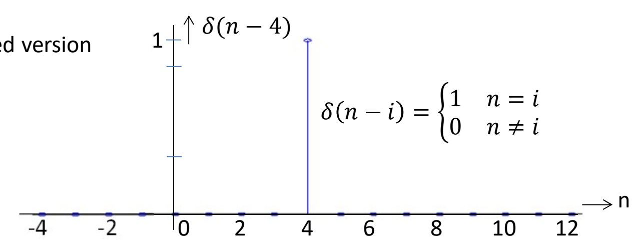
Important signals

1. The unit impulse $\delta(n)$



$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases}$$

Shifted version



$$\delta(n-i) = \begin{cases} 1 & n = i \\ 0 & n \ne i \end{cases}$$

We can write:

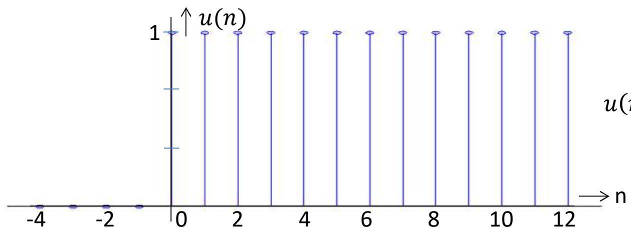
$$x(n) = \cdots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \cdots = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Special form of convolution: $x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$

8

1.9

2. The unit step function $u(n)$



$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$u(n) = \sum_{k=-\infty}^{\infty} u(k)\delta(n-k) = \sum_{k=0}^{\infty} \delta(n-k)$$

9

1.10

3. The sinusoidal signal

$x(n) = Asin(n\theta + \varphi)$

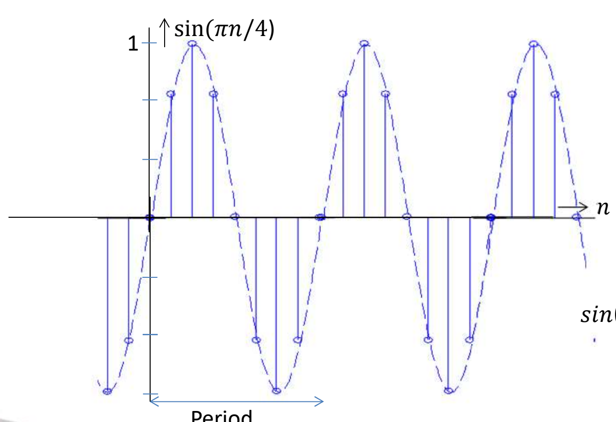
A is amplitude

θ is relative frequency

φ is phase

Example: A=1; $\theta = {}^{\pi}/_{4}$ and $\varphi = 0$

$x(n) = sin({}^{n\pi}/_{4})$



sin(πn/4) is periodic with period N=8

$$sin({}^{n\pi}/_{4}) \stackrel{?}{=} sin[\pi(n+8)/4] = \sin\left(\frac{\pi n}{4} + 2\pi\right) = \sin(\frac{\pi n}{4})$$

10

1.11

A digital signal is periodic of some period N; if N is the smallest integer for which

$x(n+N) = x(n)$ for all n

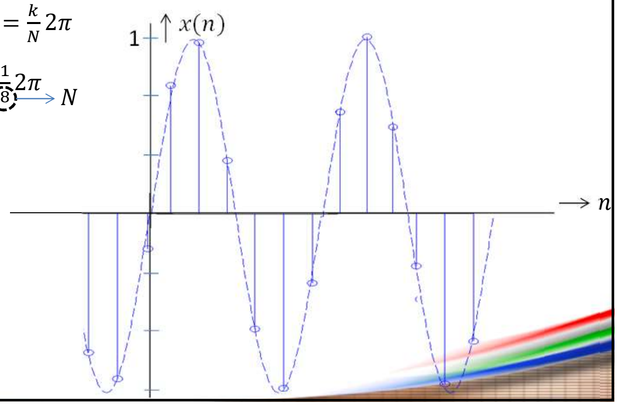
If x(n) is periodic, it must be of infinite duration (except $x(n) \equiv 0$)

Is a sinusoidal signal periodic? $A\ sin(n\theta + \varphi) \stackrel{?}{=} A\ sin[(n+N)\theta + \varphi] =$ A $\sin(n\theta + \varphi + N\theta)$

Only if: $N\theta = k2\pi$ or $\theta = \frac{k}{N} 2\pi$

Previous example: $\theta = \frac{\pi}{4} = \frac{1}{8} 2\pi$ (8 → N)

Example: $x(n) = \sin(n)$



11

1.12

Consider three sinusoidal signals: $\theta_1 = \theta$ $\theta_2 = 2\pi - \theta$ $\theta_3 = \theta + 2\pi$

With $0 \leq \theta \leq \pi$

$x_1(n) = A \sin(n\theta_1) = A \sin(n\theta)$

$x_2(n) = A \sin(n\theta_2) = A \sin\big(n(2\pi - \theta)\big) = A \sin(n2\pi - n\theta) = A \sin(-n\theta) = -A \sin(n\theta)$

$x_3(n) = A \sin(n\theta_3) = A \sin\big(n(2\pi + \theta)\big) = A \sin(n2\pi + n\theta) = A \sin(n\theta)$

We see: $x_1(n) = -x_2(n) = x_3(n)$ (They are indistinguishable)

For every θ outside the interval $[0, \pi]$ there corresponds a θ' inside $[0, \pi]$ which gives the same time function.

The relative frequency θ is always in the interval $0 \leq \theta \leq \pi$

12

Frequency description **1.13**

Survey signal analysis page 1.6:

θ is the normalized frequency

The Fourier transformation for a discrete signal is:

$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta} \quad \text{(DFT)}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)e^{jn\theta}d\theta \quad \text{(IFTD)}$$

Remarks:

1. Alternatives: $X(e^{j\theta})$; $X(e^{j\omega})$; $X^*(j\omega)$
2. $X(\theta)$ is a complex function:

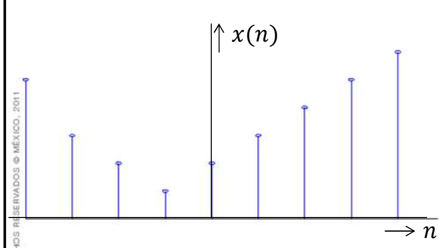
   $X = R + jI = Ae^{j\Psi}$ $\quad R, I, A$ and Ψ are functions of θ
3. θ is a relative frequency $\theta = \omega T$
4. DFT exists if $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$
5. $X(\theta)$ is periodic with period $\theta = 2\pi$

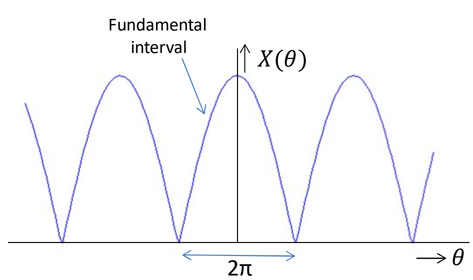
13

Proof $X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta}$ **1.14**

$$X(\theta + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn(\theta+2\pi k)} = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta}e^{-j} = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta} = X(\theta)$$

Therefore: $X(\theta)$=$X(\theta + 2\pi k)$





We have seen: $\sin(n\,\theta)$ assume $0 \le \theta \le \pi$

But: $\sin(n\theta) = \frac{e^{jn\theta} - e^{-jn\theta}}{2j}$ $\quad$ Therefore: for $e^{j\theta n}$ assume $-\pi \le \theta \le \pi$

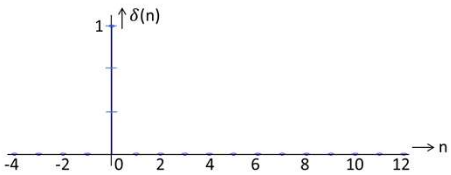
14

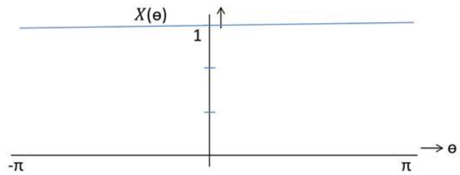
Examples

1.15

1. $x(n) = \delta(n)$

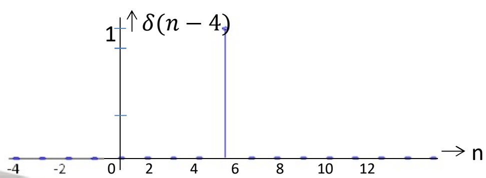
$$X(\theta) = \sum_{n=-\infty}^{\infty} \delta(n)e^{-jn\theta} = e^{-j\theta 0} = 1$$

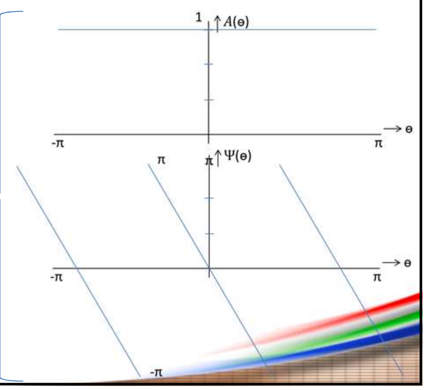




2. $x(n) = \delta(n-i)$

$$X(\theta) = \sum_{n=-\infty}^{\infty} \delta(n-i)e^{-jn\theta} = e^{-j\theta i}$$





1.16

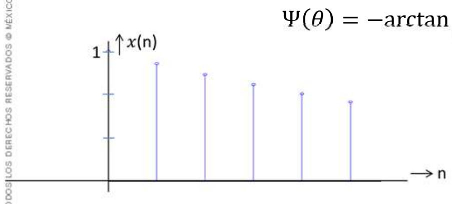
3. $x(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad |a| < 1$

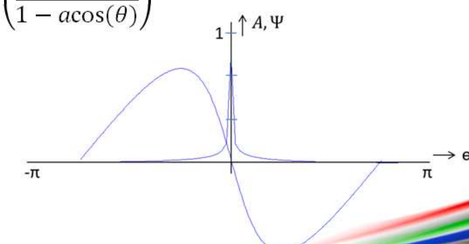
$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta} = \sum_{n=0}^{\infty} a^n e^{-jn\theta} = \sum_{n=0}^{\infty} \left(ae^{-j}\ \right)^n \qquad for\ |a| < 1$$

$$= \frac{1}{1 - ae^{-j\theta}} = \frac{1}{1 - a\cos(\theta) + aj\sin(\theta)}$$

$$A(\theta) = \sqrt{\frac{1}{1 - 2a\cos(\theta) + a^2\cos^2(\theta) + a^2\sin^2(\theta)}} = \sqrt{\frac{1}{1 - 2a\cos(\theta) + a^2}}$$

$$\Psi(\theta) = -\arctan\left(\frac{a\sin(\theta)}{1 - a\cos(\theta)}\right)$$





Properties

1.17

1. Linearity $ah(n) + bg(n) \leftrightarrow aH(\theta) + bG(c)$

2. Time shift $h(n-i) \leftrightarrow H(\theta)e^{-j\theta i}$

3. Frequency shift $h(n)e^{jn\theta_0} \leftrightarrow H(\theta - \theta_0)$

Modulation

$$2h(n)\cos(n\theta_0) \leftrightarrow H(\theta-\theta_0) + H(\theta+\theta_0)$$

$$2h(n)\sin(n\theta_0) \leftrightarrow \frac{1}{j}[H(\theta-\theta_0) - H(\theta+\theta_0)]$$

4. Convolution

$$y(n) = h(n) * g(n) \leftrightarrow H(\theta)G(\theta) = Y(\theta)$$

$$y(n) = h(n)g(n) \leftrightarrow H(\theta) * G(\theta) = Y(\theta)$$

Remark:

$$h(n) * g(n) = \sum_{k=-\infty}^{\infty} h(k)g(n-k)$$

$$H(\theta) * G(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\zeta)G(\theta-\zeta)d\zeta$$

17

5. Relation of Parseval

1.18

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\theta)|^2 d\theta$$

6. $h(n)$ real $[h(n) = h^*(n)]$

$H(\theta) = H^*(-\theta) \leftrightarrow R(\theta) = R(-\theta)$

$I(\theta) = -I(-\theta)$

$A(\theta) = A(-\theta)$

$\Psi(\theta) = -\Psi(-\theta)$

7. $h(n)$ even $[h(n) = h\,(-n)]$

$I(\theta) = 0$

$$H(\theta) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\theta} = h(0) + \sum_{n=1}^{\infty} h(n)e^{-jn\theta} + h(-n)e^{jn}$$

$$= h(0) + \sum_{n=1}^{\infty} h(n)\left(e^{-jn\theta} + e^{jn\theta}\right) = \mathrm{h}(0) + 2\sum_{n=1}^{\infty} h(n)\cos(n\theta)$$

8. $h(n)$ odd $[h(n) = -h\,(-n)]$

$$H(\theta) = 2j\sum_{n=1}^{\infty} h(n)\sin(n\theta)$$

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Z-Transform

1.19

- Continues-time
  - FTC: Real frequencies f or ω
  - Laplace: Complex frequencies s
    - (poles/zeroes)
- Discrete/time
  - FTD: Real frequencies
    - Normalized Θ
    - Un-normalized ω
  - Complex frequencies
  - Poles/zeroes } ?

Z-Transform

- z is the complex frequency
- There is a relation between the z-transform and the FTD
- The z transform is a universal transform for digital signals and digital systems

19

Definition

1.20

$$\tilde{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \text{(ZT)}$$

- z is a complex number
- $\tilde{X}(z)$ is also complex and is defined for those values of z for which $\tilde{X}(z)$ converges

The inverse transformation is defined by:

$$x(n) = \frac{1}{2\pi j} \oint_{c} \tilde{X}(z) z^{n-1} dz \quad \text{(IZT)}$$

- C is any closed contour in the region of convergence that encloses z=0
- Fortunately the IZT is never used in practice

$$\tilde{X}(z) = \frac{T(z)}{N(z)} = \frac{A_0 + A_1 z^{-1} + \cdots}{B_0 + B_1 z^{-1} + \cdots}$$

20

1.21

Examples

1. $x(n) = \delta(n) \rightarrow \tilde{X}(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = z^{-0} = 1$

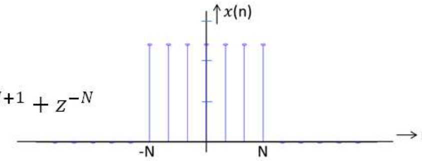
---

2. $x(n) = \delta(n-k) \rightarrow \tilde{X}(z) = \sum_{n=-\infty}^{\infty} \delta(n-k) z^{-n} = z^{-k}$

---

3. $x(n) = \begin{cases} 1 & for \quad |n| \le N \\ 0 & for \quad |n| > N \end{cases}$

$$\tilde{X}(z) = \sum_{n=-N}^{N} z^{-n} = z^{N} + z^{N-1} + \cdots + z^{-N+1} + z^{-N}$$



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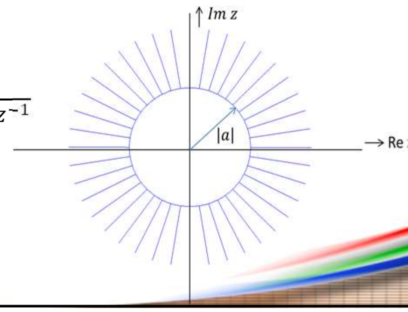
4. $x(n) = a^n u(n)$

$$\tilde{X}(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$$

$for \quad |az^{-1}| < 1$

$or \quad |z| > |a|$



21

1.22

5. $x(n) = \rho^n \cos(n\varphi)\, u(n) = \frac{1}{2}\rho^n \left[e^{jn\varphi} + e^{-jn\varphi}\right] u(n)$

$$\tilde{X}(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left[\rho^n e^{jn\varphi} + \rho^n e^{-jn\varphi}\right] z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left[\rho\, e^{j\varphi} z^{-1}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[\rho\, e^{-j\varphi} z^{-1}\right]^n$$

$$= \frac{1}{2}\left[\frac{1}{1 - \rho e^{j\varphi} z^{-1}} + \frac{1}{1 - \rho e^{-j\varphi} z^{-1}}\right] = \frac{1}{2}\left[\frac{1 + 1 - \rho z^{-1}\left(e^{j\varphi} + e^{-j\varphi}\right)}{1 - \rho z^{-1}\left(e^{j\varphi} + e^{-j\varphi}\right) + \rho^2 z^{-2}}\right]$$

$$\tilde{X}(z) = \frac{1 - \rho z^{-1}\cos(\varphi)}{1 - 2\rho z^{-1}\cos(\varphi) + \rho^2 z^{-2}}$$

---

6. $x(n) = \rho^n \sin(n\varphi)\, u(n)$

$$\tilde{X}(z) = \frac{\rho z^{-1}\sin(\varphi)}{1 - 2\rho z^{-1}\cos(\varphi) + \rho^2 z^{-2}}$$

22

1.23

$$\rho^n \cos(n\varphi)\, u(n) \leftrightarrow \frac{1 - \rho z^{-1}\cos(\varphi)}{1 - 2\rho z^{-1}\cos(\varphi) + \rho^2 z^{-2}}$$

$$\rho^n \sin(n\varphi)\, u(n) \leftrightarrow \frac{\rho z^{-1}\sin(\varphi)}{1 - 2\rho z^{-1}\cos(\varphi) + \rho^2 z^{-2}}$$

$$\tilde{X}(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1} + b_2 z^{-2}} \leftrightarrow x(n) = ?$$

Example

$$\tilde{X}(z) = \frac{2 + \frac{1}{6} z^{-1}}{1 - \frac{1}{3} z^{-1} + \frac{1}{9} z^{-2}} \leftrightarrow [A\rho^n \cos(n\varphi) + B\rho^n \sin(n\varphi)]u(n)$$

$$= \frac{A + (B\rho\, \sin(\varphi) - A\rho\, \cos(\varphi))z^{-1}}{1 - 2\rho z^{-1}\cos(\varphi) + \rho^2 z^{-2}}$$

$$A = 2$$
$$B\rho\, \sin(\varphi) - A\rho\, \cos(\varphi) = \frac{1}{6}$$
$$2\rho\, \cos(\varphi) = \frac{1}{3}$$
$$\rho^2 = \frac{1}{9} \rightarrow \rho = \frac{1}{3}$$

$$\frac{1}{3} B \sin(\varphi) - \frac{2}{3}\cos(\varphi) = \frac{1}{6}$$
$$\frac{2}{3}\cos(\varphi) = \frac{1}{3} \rightarrow \varphi = \frac{\pi}{3}$$
$$\frac{1}{3} B \sin(\varphi) - \frac{2}{3}\cos(\varphi) = \frac{B}{2\sqrt{3}} - \frac{1}{3} = \frac{1}{6} \rightarrow B = \sqrt{3}$$

$$\tilde{X}(z) = \frac{2 + \frac{1}{6} z^{-1}}{1 + \frac{1}{3} z^{-1} + \frac{1}{9} z^{-2}} \leftrightarrow x(n) = \left[2\left(\frac{1}{3}\right)^n \cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\left(\frac{1}{3}\right)^n \sin\left(\frac{n\pi}{3}\right)\right] u(n)$$

23

1.24

Inverse transformation

1. Use IZT formula
2. Partial fraction expansion

$$\tilde{X}(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{j=1}^{M} b_j z^{-j}} = \sum_{i=1}^{I} \frac{\alpha_i}{1 - \beta_i z^{-1}} + \sum_{j=1}^{J} \frac{a_{0j} + a_{1j} z^{-1}}{1 - b_{1j} z^{-1} + b_{2j} z^{-2}}$$

$$x(n) = \sum_{i=1}^{I} \alpha_i (\beta_i)^n u(n) + \sum_{j=1}^{J} \left(A_j \cos(n\varphi_j) + B_j \sin(n\varphi_j)\right)\rho^n u(n)$$

Remark: if $N \geq M$ there are additional terms

$$\ldots + \sum_{k=1}^{N-M} c_k z^{-k} \rightarrow \cdots + \sum_{k=1}^{N-M} c_k \delta(n-k)$$

3. Ordinary division

$$\tilde{X}(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots = \sum_{k=0}^{\infty} a_k z^{-k}$$

$$x(n) = a_0 \delta(n) + a_1 \delta(n-1) + a_2 \delta(n-2) + \cdots = \sum_{k=0}^{\infty} a_k \delta(n-k)$$

24

Example 1.25

$$\tilde{X}(z)=\frac{9+21z^{-1}+32z^{-2}}{1+4z^{-1}+9z^{-2}+6z^{-3}}=\frac{9+21z^{-1}+32z^{-2}}{(1+z^{-1})(1+3z^{-1}+6z^{-2})}$$

$$=\frac{A+Bz^{-1}}{1+3z^{-1}+6z^{-2}}+\frac{C}{1+z^{-1}}=\frac{(A+Bz^{-1})(1+z^{-1})+C(1+3z^{-1}+6z^{-2})}{(1+z^{-1})(1+3z^{-1}+6z^{-2})}$$

$$=\frac{(A+C)+(A+B+3C)z^{-1}+(B+6C)z^{-2}}{(1+z^{-1})(1+3z^{-1}+6z^{-2})}$$

We see:
A+C=9
A+B+3C=21
B+6C=32
} A=4; B=2; C=5

$$\tilde{X}(z)=\frac{4+2z^{-1}}{1+3z^{-1}+6z^{-2}}+\frac{5}{1+z^{-1}}\Longrightarrow x(n)$$

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Examples 1.26

$$a.\ \tilde{X}(z)=\frac{1-z^{-1}-5z^{-2}-3z^{-3}}{1-3z^{-1}}$$

Divide: $1-3z^{-1}$ ⟌ $1-z^{-1}-5z^{-2}-3z^{-3}$ → $1+2z^{-1}+z^{-2}$

$1-3z^{-1}$ -

$2z^{-1}-5z^{-2}-3z^{-3}$

$2z^{-1}-6z^{-2}$ -

$z^{-2}-3z^{-3}$

$z^{-2}-3z^{-3}$ -

0

$$\tilde{X}(z)=1+2z^{-1}+z^{-2}\rightarrow x(n)=\delta(n)+2\delta(n-1)+\delta(n-2)$$

$$b.\ \tilde{X}(z)=\frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\cdots$$

$$x(n)=\delta(n)+a\delta(n-1)+a^2\delta(n-2)+\cdots=a^nu(n)$$

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Properties of the z-transform

1.27

1. Linearity

$$ah(n) + bg(n) \leftrightarrow a\tilde{H}(z) + b\tilde{G}(z)$$

2. Time shift

$$h(n-i) \leftrightarrow z^{-i}\tilde{H}(z)$$

3. Scaling

$$a^n h(n) \leftrightarrow \tilde{H}(z/a)$$

4. Convolution

$$h(n) * g(n) \leftrightarrow \tilde{H}(z)\tilde{G}(z)$$

Where: $h(n) * g(n) = \sum_{i=-\infty}^{\infty} h(i)g(n-i)$

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Relation between ZT and DFT

1.28

$$\tilde{X}(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad \text{(ZT)}$$

$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta} \qquad \text{(FTD)}$$

Now substitute $z = e^{j\theta}$ into the ZT;

$$\tilde{X}(z)\Big|_{z=e^{j\theta}} : \tilde{X}(e^{j\theta}) = X(\theta)$$

The fundamental interval $-\pi \le \theta \le \pi$ of X(θ) corresponds to the unit circle $|z| = 1$ in the z-domain

θ	$z = e^{j\theta}$
-π	$e^{-j\pi} = -1$
-π/2	$e^{-j\pi/2} = -j$
0	$e^{j0} = 1$
π/2	$e^{j\pi/2} = j$
π	$e^{j\pi} = -1$

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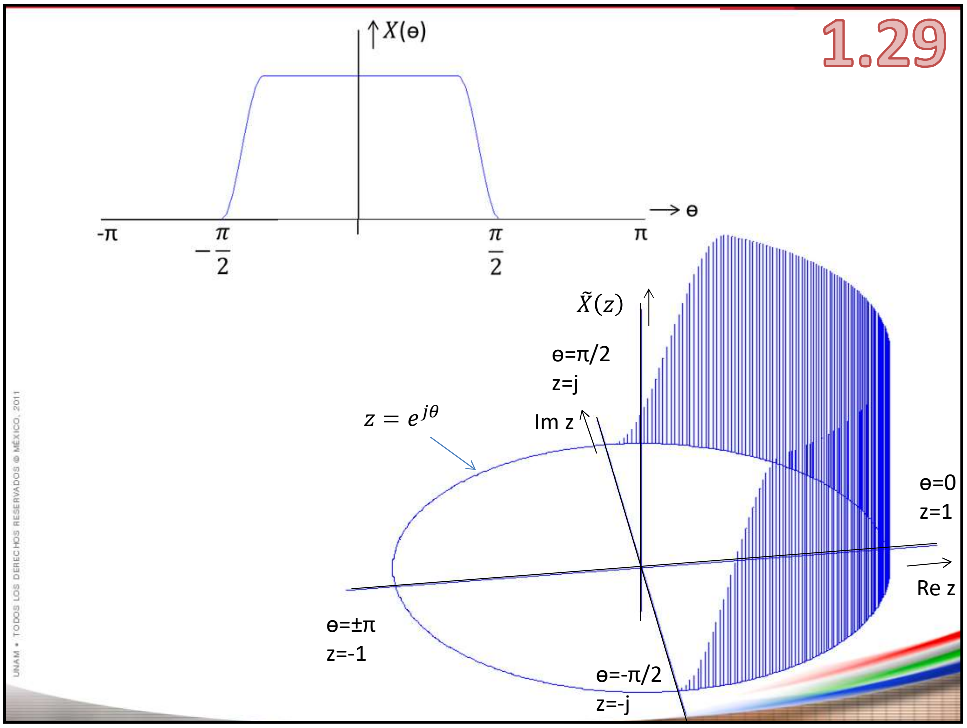

1.29
$X(\theta)$
θ
-π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\tilde{X}(z)$
θ=π/2
z=j
$z = e^{j\theta}$
Im z
θ=0
z=1
Re z
θ=±π
z=-1
θ=-π/2
z=-j
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