4.1

Digital filters

Classifications:

- Structure

1. Non recursive Digital Filters (NRDF)
2. Recursive Digital Filters (RDF)

- Impulse response

3. Finite impulse Response Digital Filters (FIR)
4. Infinite impulse Response Digital Filters (IIR)

Relations:

$$\begin{tikzcd} \text{NRDF} \arrow[r] & \text{FIR} \\ \text{RDF} \arrow[r] \arrow[ur] & \text{IIR} \end{tikzcd}$$

This means:

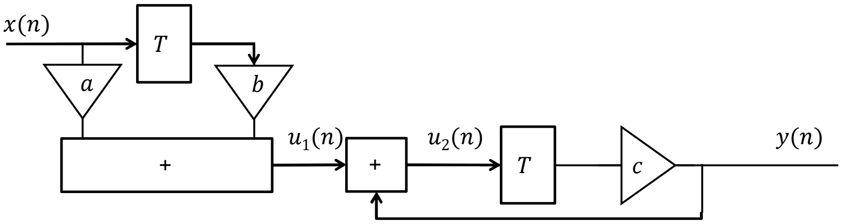
NRDF	always FIR
IIR	always RDF
FIR	both NRDF and RDF
RDF	both FIR and IIR

1. NRDF: Each digital filter can be described by a set of difference equations.

Variables:
x(n) input signal
y(n) output signal
$u_1(n)$, $u_2(n)$,… internal variable

4.2

Example



$$\left.\begin{aligned} u_1(n) &= ax(n) + bx(n-1) \\ u_2(n) &= u_1(n) + y(n) \\ y(n) &= cu_2(n-1) \end{aligned}\right\} \quad y(n) \;= cu_1(n-1) + cy(n-1)$$

A set of difference equations is called non recursive if none of the $u_i(n)$ or $y(n)$ depends on previous values of itself (no feedback)

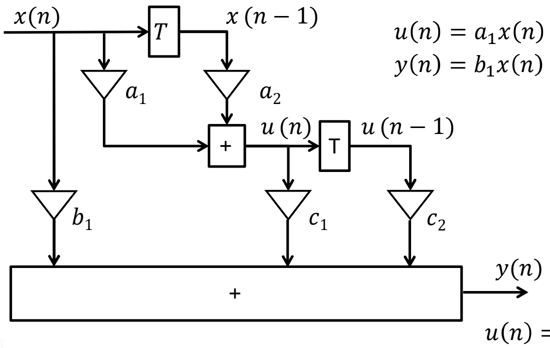
Therefore: The example is a recursive structure

Remark: In the survey a none theoretical treatment is given

→

Example

4.3



$$u(n) = a_1 x(n) + a_2 x(n-1) \quad (1)$$

$$y(n) = b_1 x(n) + c_1 u(n) + c_2 u(n-1) \quad (2)$$

Special form: transversal filter

From (1):

$$u(n) = a_1 x(n) + a_2 x(n-1) \quad (3)$$

$$u(n-1) = a_1 x(n-1) + a_2 x(n-2) \quad (4)$$

Substitute (3) and (4) into (2):

$$y(n) = b_1 x(n) + c_1 a_1 x(n) + c_1 a_2 x(n-1) + c_2 a_1 x(n-1) + c_2 a_2 x(n-2)$$

$$= (b_1 + c_1 a_1) x(n) + (c_1 a_2 + c_2 a_1) x(n-1) + c_2 a_2 x(n-2)$$

$$= A_1\, x(n) + A_2 x(n-1) + A_3 x(n-2)$$

General: $y(n) = \sum_{i=0}^{M} A_i x(n-i)$ $\quad \mathrm{h(n)} = \begin{cases} 0 & for \quad n < 0 \quad and \quad n > M \\ A_M & for \quad 0 \le n \le M \end{cases}$

See section on FIR filters

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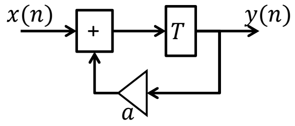
2. RDF

4.4

- In an RDF at least one of the variables depends on previous values of itself
- Any RDF must contain at least one closed loop
- Realizability: each closed loop contains at least one delay element

Example 1



$$y(n) = x(n-1) + a y(n-1)$$

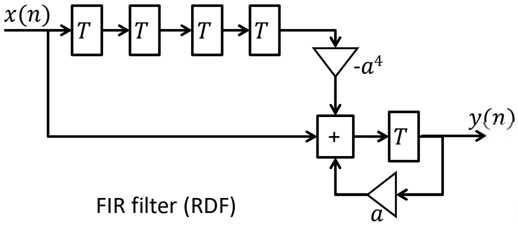
$$h(n) = \begin{cases} 0 & n \le 0 \\ a^{n-1} & n \ge 1 \end{cases}$$

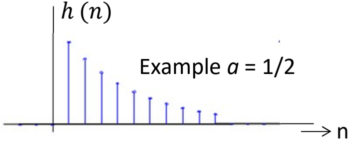
IIR filter (RDF)

Example 2

$$y(n) = -a^4 x(n-5) + a y(n-1)$$

$$h(n) = \begin{cases} 0 & n \le 0 \\ a^{n-1} & 1 \le n \le 4 \\ 0 & n \ge 5 \end{cases}$$





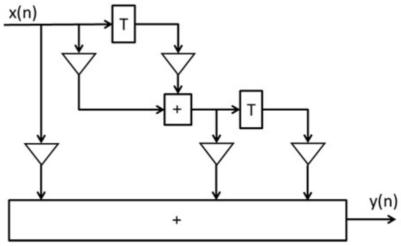
FIR filter (RDF)

Remark: exact cancelation from “forward” and “feedback” path.

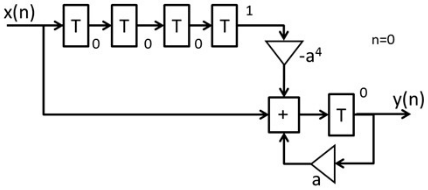
Initial conditions

Until now it was assumed that a digital filter is started with empty registers with initial conditions:

- For a NRDF the influence of the initial conditions has disappeared after at most N periods ($N = length\ of\ h(n)$)



- For a RDF an addition response results
  - If the filter is stable, the transient response disappears
  - Even for a FIR filter the transient may have an infinite duration



$$x(n) \equiv 0 \rightarrow y(n) = -a^4\delta(n-1) - a^5\delta(n-2) - \cdots = \begin{cases} 0 & n \leq 0 \\ -a^{n-5} & n \geq 1 \end{cases}$$

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3. FIR filter

A causal FIR filter of "length" N+1:

- $h(n) = 0\ for\ n < 0\ and\ n > N$
- Is stable because $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{N} |h(n)| \quad is\ finite$

Transmission function: $H(\theta) = \sum_{n=0}^{N} h(n)e^{-jn\theta}$

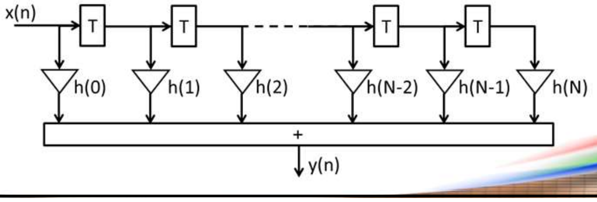
System function:

$$\tilde{H}(z) = \sum_{n=0}^{N} h(n)z^{-n} = h(0) + h(1)z^{-1} + \cdots + h(N)z^{-N}$$

$$= \frac{h(0)z^N + h(1)z^{N-1} + \cdots + h(N)}{z^N} = \frac{h(0)}{z^N}(z - z_1)(z - z_2)\ldots(z - z_N)$$

N zeros in the z-plane, N poles in z=0 (thus filter is stable)

Transversal filter
or
direct form structure



4.7

Transpose direct form

Define $u_0(n) \dots u_N(n)$ $\qquad x(n) = \delta(n)$

$$u_0(n) = h(N)x(n) \qquad = h(N)\delta(n)$$

$$u_1(n) = h(N-1)x(n) + u_0(n-1) \qquad = h(N-1)\delta(n) + h(N)\delta(n-1)$$

$$u_2(n) = h(N-2)x(n) + u_1(n-1) \qquad = h(N-2)\delta(n) + h(N-1)\delta(n-1) + h(N)\delta(n-2)$$

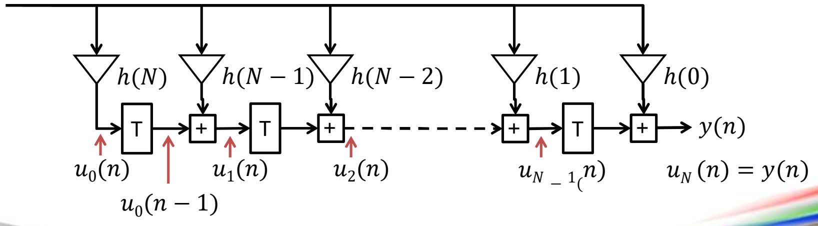
$$\vdots \qquad \vdots$$

$$u_k(n) = h(N-k)x(n) + u_{k-1}(n-1) \qquad = h(N-k)\delta(n) + h(N-k+1)\delta(n-1) + \cdots + h(N)\delta(n-k)$$

$$\vdots \qquad \vdots$$

$$u_N(n) = h(0)x(n) + u_{N-1}(n-1) \qquad = h(0)\delta(n) + h(1)\delta(n-1) + \cdots + h(N)\delta(n-N)$$

$$y(n) = u_N(n) \qquad = y(n) = \sum_{k=0}^{N} h(k)\delta(n-k) = h(n)$$



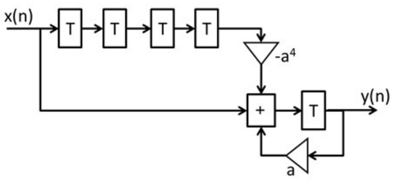
4.8

Both structures use:
- N registers (less than N is impossible)
  They are called canonic realizations

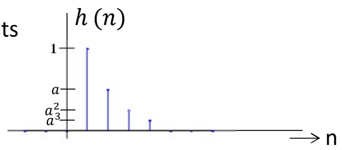
Both structures require:
- N+1 multiplications
- N additions

} Computationally equivalent

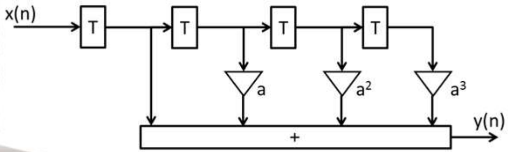
Example



5 delay elements
2 multipliers



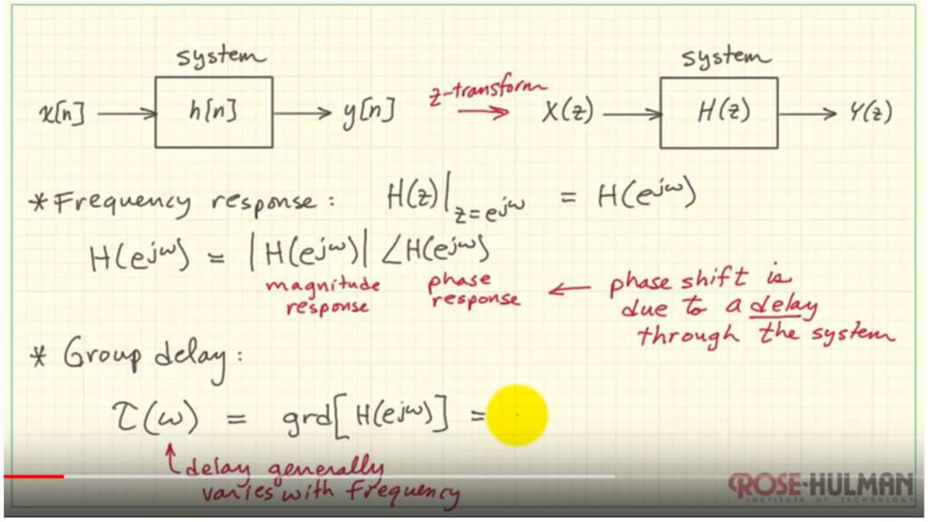
2 multiplications + 2 additions per y(n)



4 delay elements
3 multipliers

3 multiplications + 3 additions per y(n)

Group Delay



system system

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \xrightarrow{z\text{-transform}} X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z)$$

* Frequency response: $H(z)\big|_{z=e^{j\omega}} = H(e^{j\omega})$

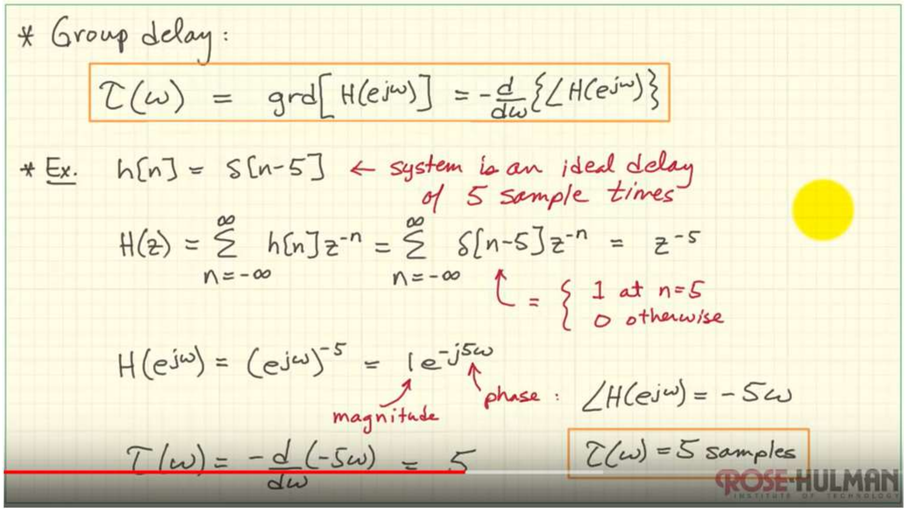
$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{magnitude response}} \underbrace{\angle H(e^{j\omega})}_{\text{phase response}}$$

← phase shift is due to a delay through the system

* Group delay:

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] =$$

↑delay generally varies with frequency



* Group delay:

$$\boxed{\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}}$$

* Ex. $h[n] = \delta[n-5]$ ← system is an ideal delay of 5 sample times

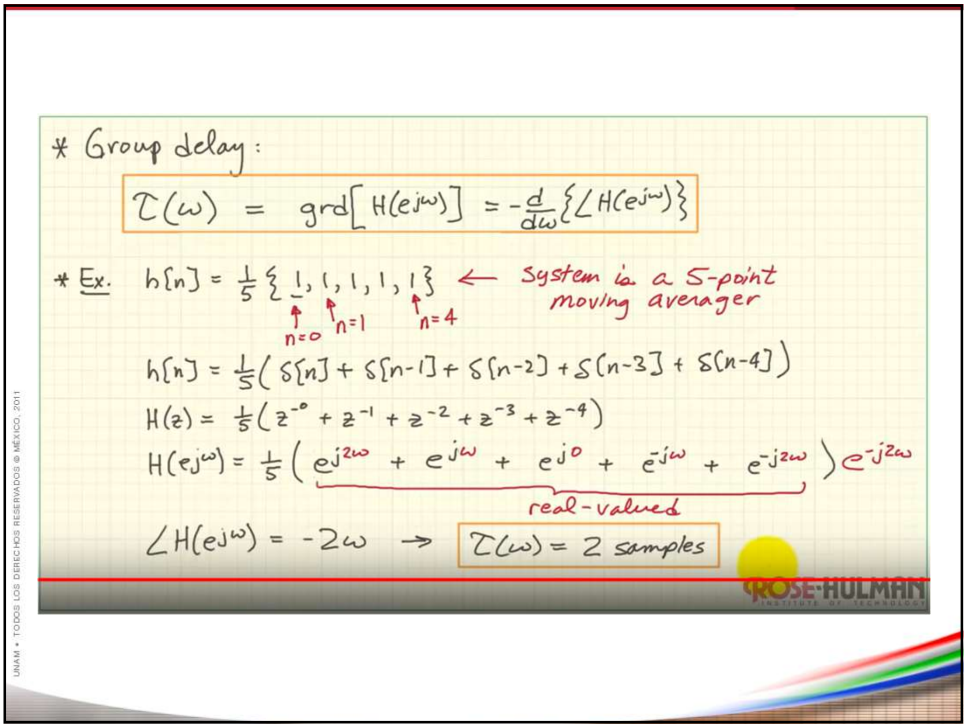
$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-5]z^{-n} = z^{-5}$$

↑ = $\begin{cases} 1 & \text{at } n=5 \\ 0 & \text{otherwise} \end{cases}$

$$H(e^{j\omega}) = (e^{j\omega})^{-5} = \underset{\text{magnitude}}{1}\, e^{-j5\omega}$$

↑ phase : $\angle H(e^{j\omega}) = -5\omega$

$$\tau(\omega) = -\frac{d}{d\omega}(-5\omega) = 5 \qquad \boxed{\tau(\omega) = 5 \text{ samples}}$$



* Group delay:

$$\boxed{\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}}$$

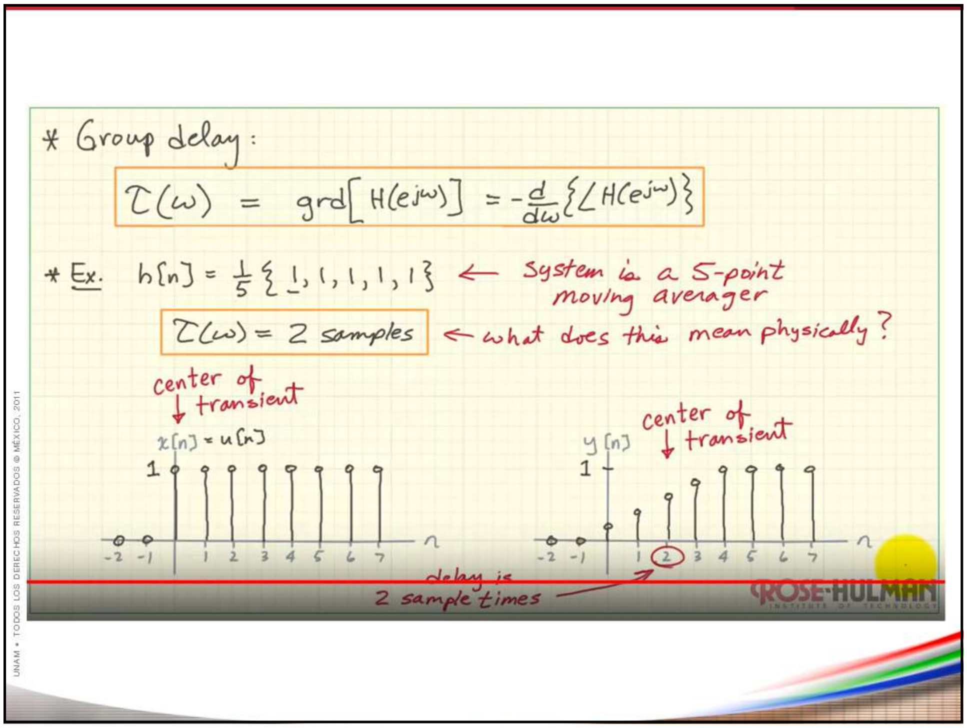
* Ex. $h[n] = \frac{1}{5}\{\underset{\uparrow n=0}{\underline{1}}, \underset{\uparrow n=1}{1}, 1, 1, \underset{\uparrow n=4}{1}\}$ ← system is a 5-point moving averager

$h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$

$H(z) = \frac{1}{5}(z^{-0} + z^{-1} + z^{-2} + z^{-3} + z^{-4})$

$H(e^{j\omega}) = \frac{1}{5}(\underbrace{e^{j2\omega} + e^{j\omega} + e^{j0} + e^{-j\omega} + e^{-j2\omega}}_{\text{real-valued}})e^{-j2\omega}$

$\angle H(e^{j\omega}) = -2\omega \rightarrow \boxed{\tau(\omega) = 2 \text{ samples}}$



* Group delay:

$$\boxed{\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}}$$

* Ex. $h[n] = \frac{1}{5}\{\underline{1}, 1, 1, 1, 1\}$ ← system is a 5-point moving averager

$\boxed{\tau(\omega) = 2 \text{ samples}}$ ← what does this mean physically?

Important property of FIR filters

4.9

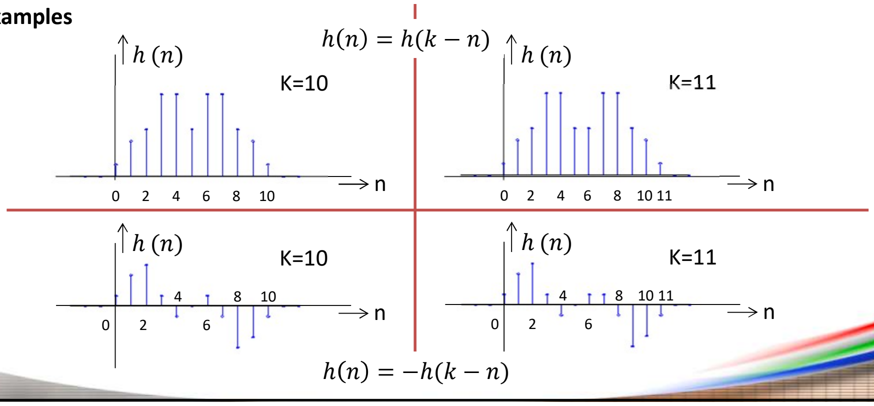
It is possible to have a linear phase $\Psi(\theta)$; this means a constant group delay $\tau_g = -{}^{d\Psi(\theta)}/_{d\theta}$

A FIR filter has a linear phase if for an integer k and for all n:

$$h(n) = h(k-n) \quad or \quad h(n) = -h(k-n)$$

Causality: $h(n) = 0 \quad for \quad n \le 0$
Therefore: $h(n) = 0 \quad for \quad n > k$

Examples



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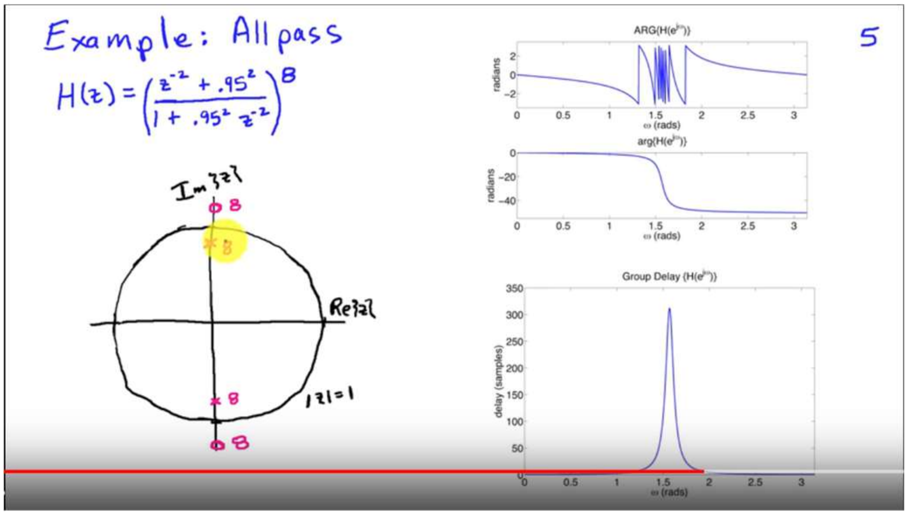

Example: All pass

$H(z) = \left(\frac{z^{-2} + .95^2}{1 + .95^2 z^{-2}}\right)^8$

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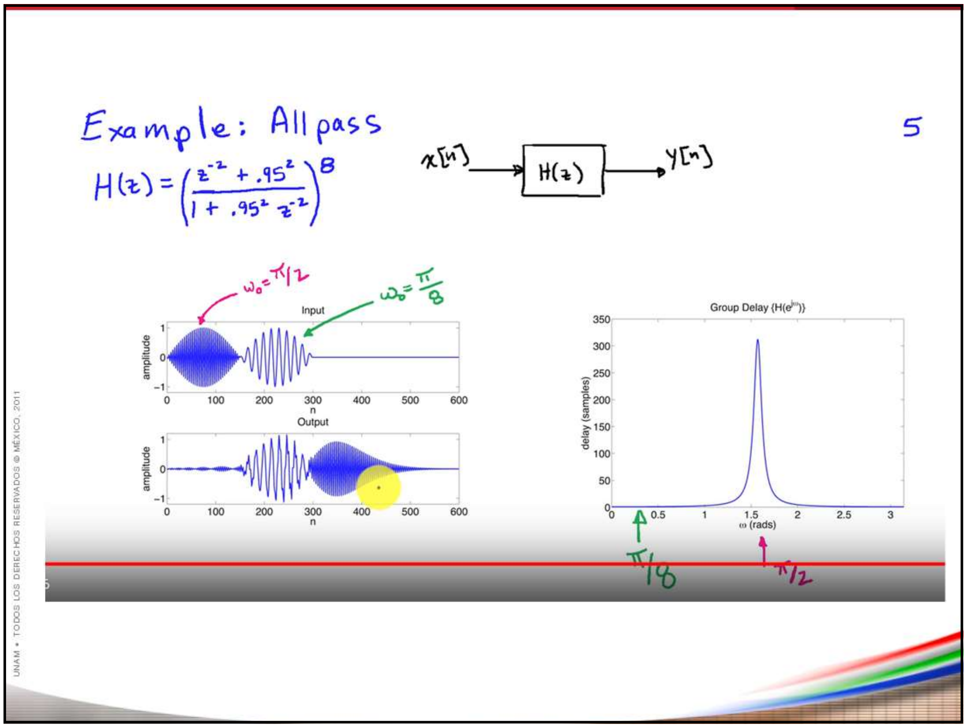


Example: All pass

5

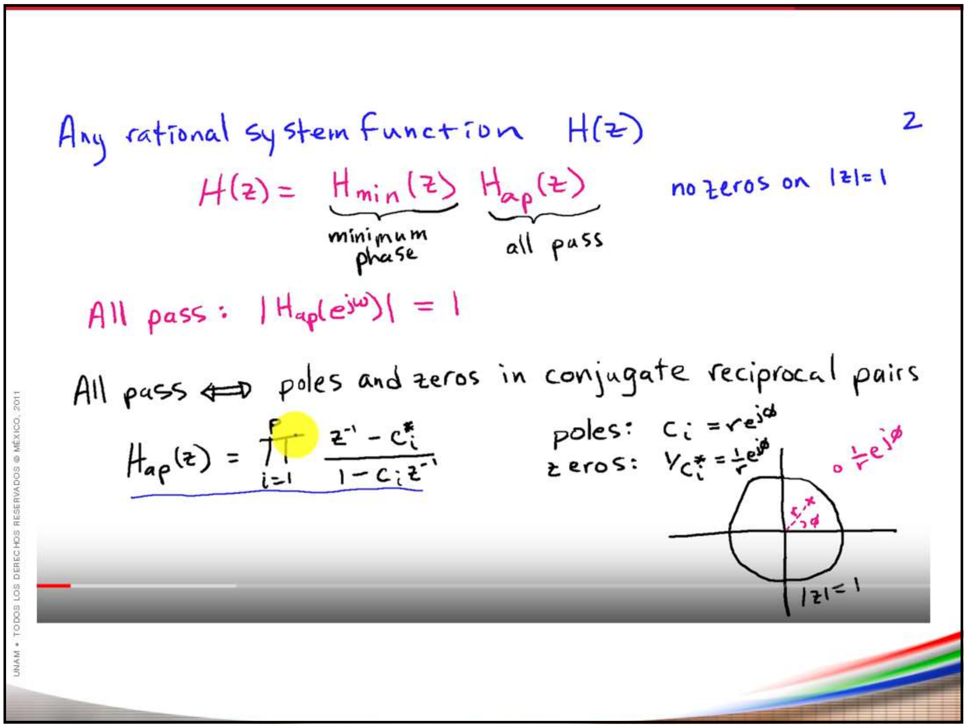
$$H(z) = \left(\frac{z^{-2} + .95^2}{1 + .95^2 z^{-2}}\right)^8$$

$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$

A stable, causal system has a stable, causal inverse
if and only if
all poles and zeros are inside $|z|=1$

called: Minimum phase system

Can show that phase lag of a system with poles/zeros inside $|z|=1$ is less than that of any other system with identical magnitude response



Any rational system function $H(z)$ 2

$$H(z) = \underbrace{H_{min}(z)}_{\text{minimum phase}} \underbrace{H_{ap}(z)}_{\text{all pass}}$$ no zeros on $|z|=1$

All pass : $|H_{ap}(e^{j\omega})| = 1$

All pass $\Longleftrightarrow$ poles and zeros in conjugate reciprocal pairs

$$\underline{H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}}$$

poles: $c_i = re^{j\phi}$

zeros: $1/c_i^* = \frac{1}{r}e^{j\phi}$

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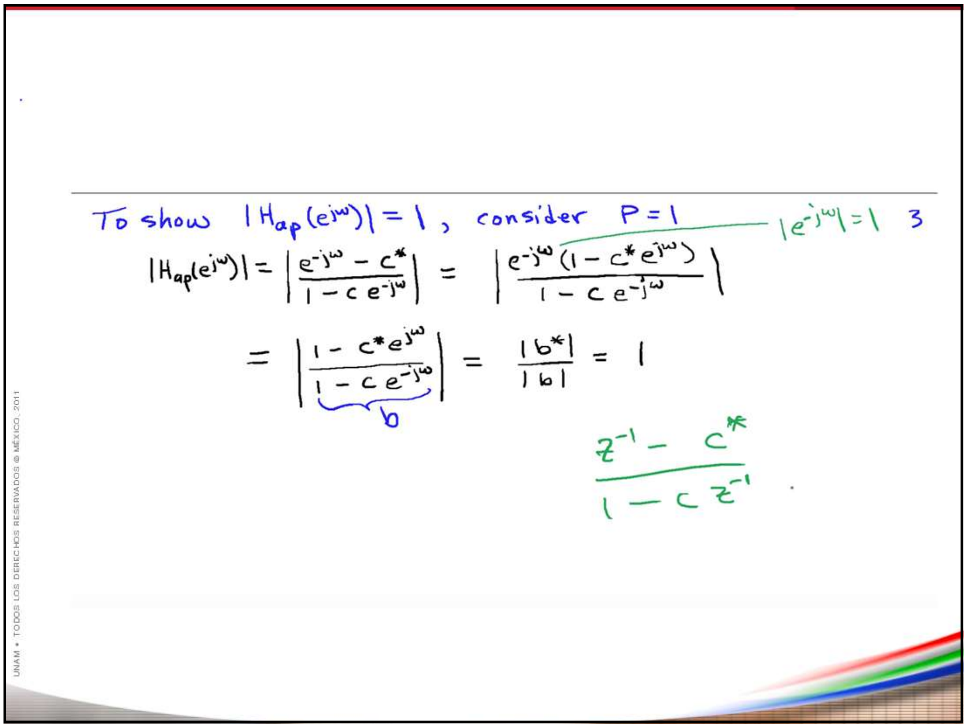

To show $|H_{ap}(e^{j\omega})| = 1$, consider $P=1$ 3

$|e^{-j\omega}| = 1$

$$|H_{ap}(e^{j\omega})| = \left|\frac{e^{-j\omega} - c^*}{1 - c\,e^{-j\omega}}\right| = \left|\frac{e^{-j\omega}(1 - c^* e^{j\omega})}{1 - c\,e^{-j\omega}}\right|$$

$$= \left|\frac{1 - c^* e^{j\omega}}{\underbrace{1 - c\,e^{-j\omega}}_{b}}\right| = \frac{|b^*|}{|b|} = 1$$

$$\frac{z^{-1} - c^*}{1 - c\,z^{-1}}$$

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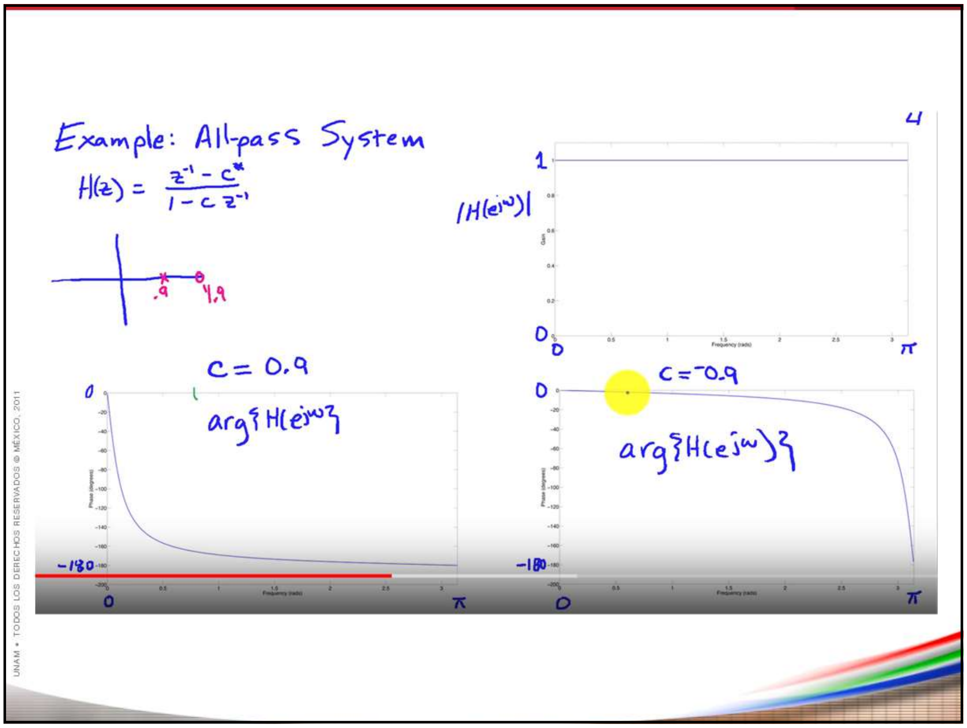


Example: All-pass System

$H(z) = \frac{z^{-1} - c^*}{1 - c z^{-1}}$

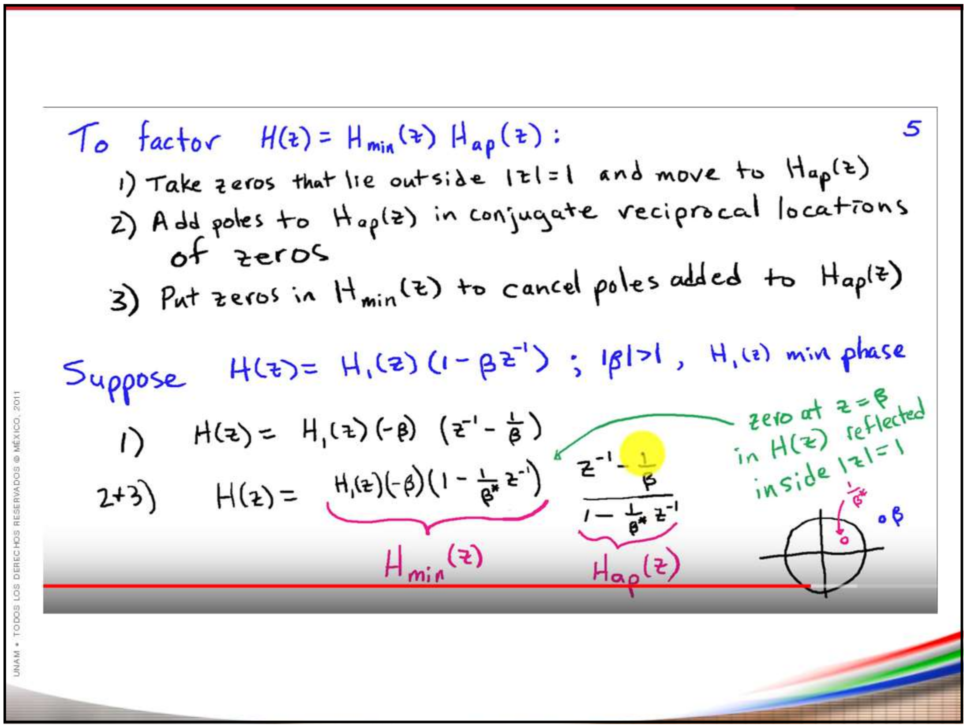
c = 0.9

$\arg\{H(e^{j\omega})\}$

c = -0.9

$\arg\{H(e^{j\omega})\}$

$|H(e^{j\omega})|$



To factor $H(z) = H_{min}(z)\, H_{ap}(z)$:

1) Take zeros that lie outside $|z|=1$ and move to $H_{ap}(z)$
2) Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of zeros
3) Put zeros in $H_{min}(z)$ to cancel poles added to $H_{ap}(z)$

Suppose $H(z) = H_1(z)(1 - \beta z^{-1})$; $|\beta| > 1$, $H_1(z)$ min phase

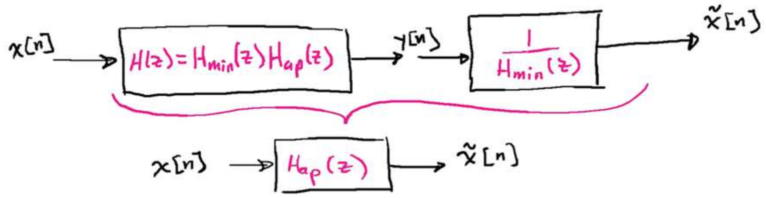
1) $H(z) = H_1(z)(-\beta)\left(z^{-1} - \frac{1}{\beta}\right)$

2+3) $H(z) = \underbrace{H_1(z)(-\beta)\left(1 - \frac{1}{\beta^*} z^{-1}\right)}_{H_{min}(z)} \underbrace{\frac{z^{-1} - \frac{1}{\beta}}{1 - \frac{1}{\beta^*} z^{-1}}}_{H_{ap}(z)}$

zero at $z = \beta$ in $H(z)$ reflected inside $|z| = 1$

The minimum phase portion of any system has a stable, causal inverse system

5

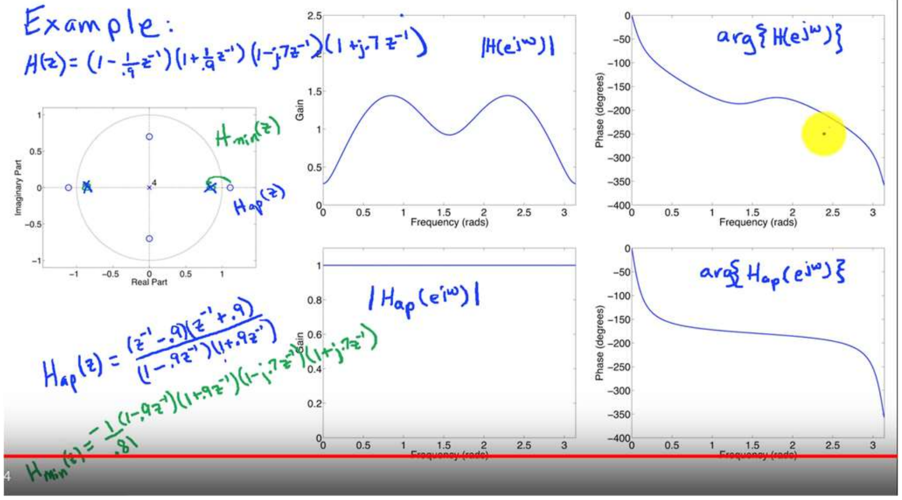


no magnitude distortion

only phase distortion

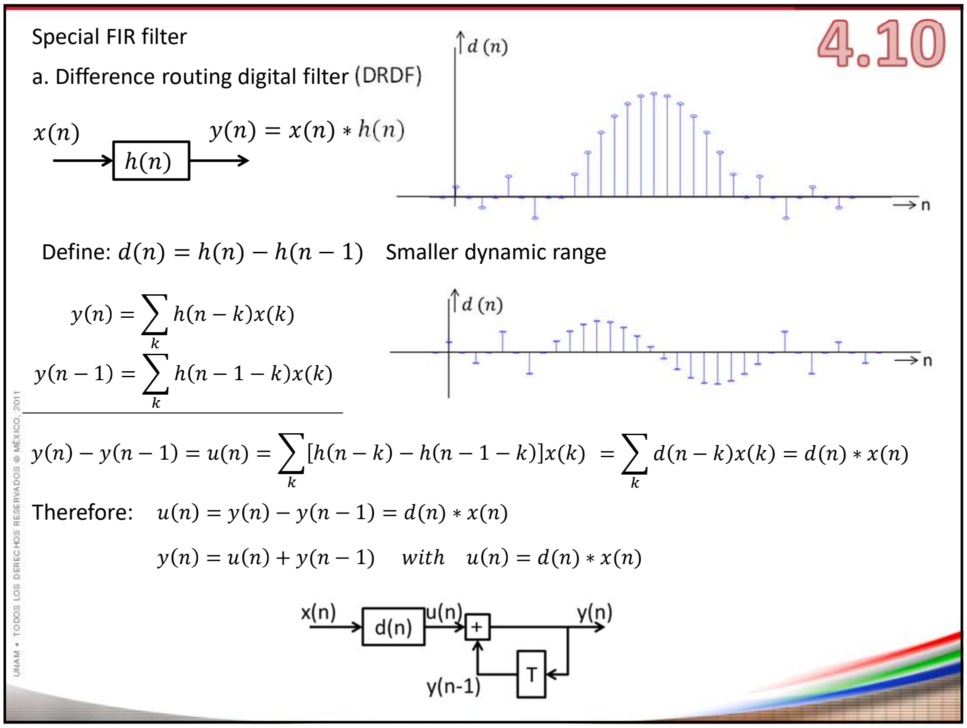
Example:

$H(z) = (1 - \frac{1}{.9}z^{-1})(1 + \frac{1}{.9}z^{-1})(1 - j.7z^{-1})(1 + j.7z^{-1})$



$$H_{ap}(z) = \frac{(z^{-1} - .9)(z^{-1} + .9)}{(1 - .9z^{-1})(1 + .9z^{-1})}$$

$$H_{min}(z) = -\frac{1}{.81}(1 - .9z^{-1})(1 + .9z^{-1})(1 - j.7z^{-1})(1 + j.7z^{-1})$$



4.10

Special FIR filter

a. Difference routing digital filter (DRDF)

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) * h(n)$$

Define: $d(n) = h(n) - h(n-1)$ Smaller dynamic range

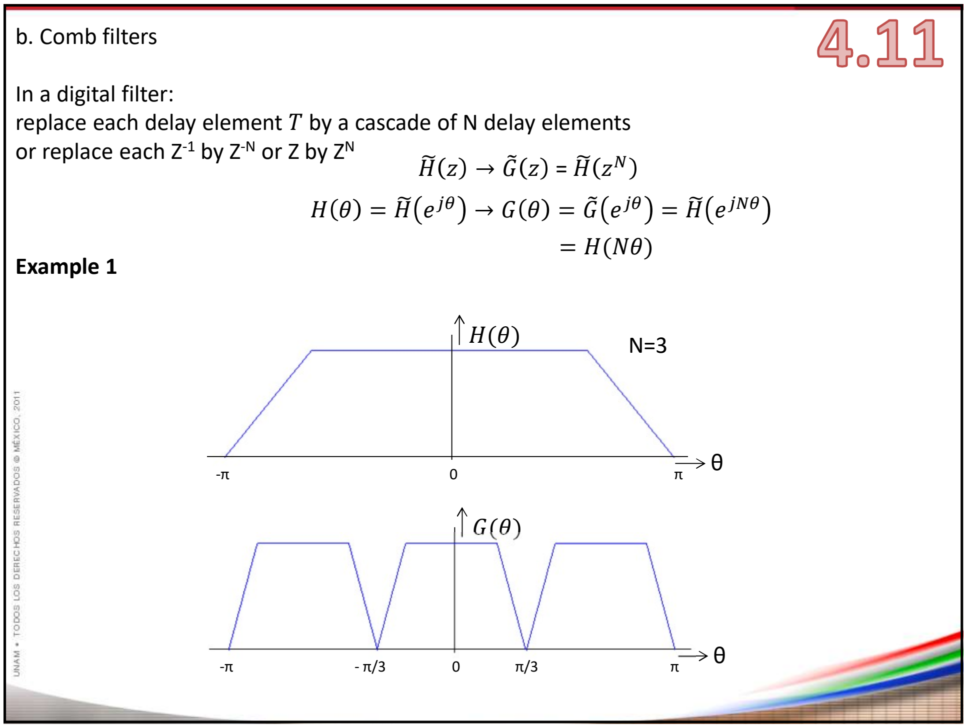
$$y(n) = \sum_k h(n-k)x(k)$$

$$y(n-1) = \sum_k h(n-1-k)x(k)$$

$$y(n) - y(n-1) = u(n) = \sum_k [h(n-k) - h(n-1-k)]x(k) = \sum_k d(n-k)x(k) = d(n) * x(n)$$

Therefore: $u(n) = y(n) - y(n-1) = d(n) * x(n)$

$$y(n) = u(n) + y(n-1) \quad with \quad u(n) = d(n) * x(n)$$



4.11

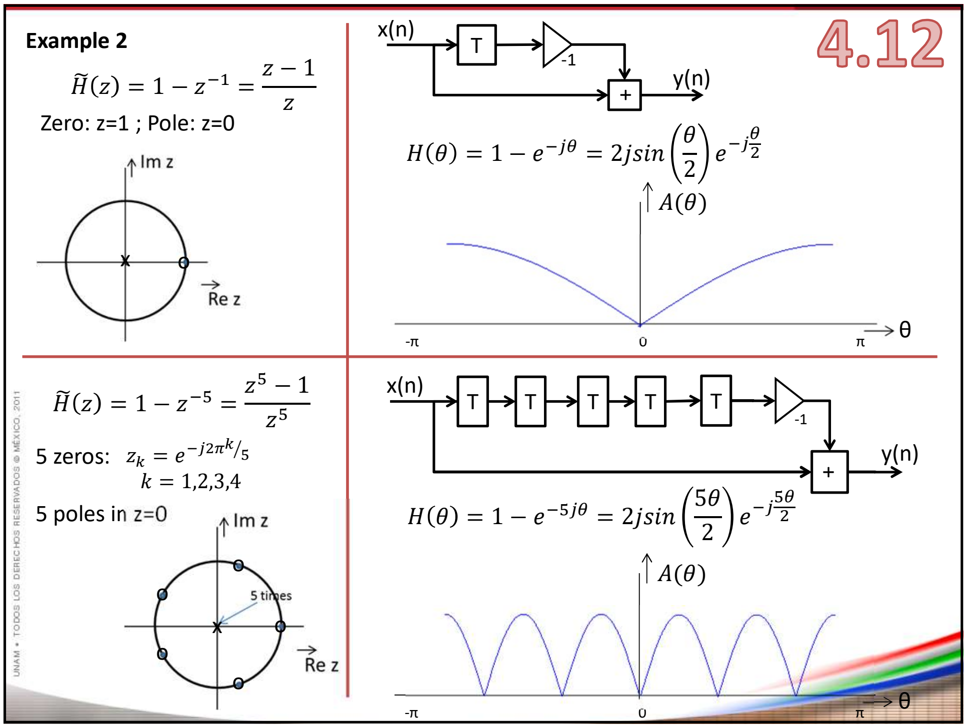
b. Comb filters

In a digital filter:
replace each delay element T by a cascade of N delay elements
or replace each Z^{-1} by Z^{-N} or Z by Z^N

$$\tilde{H}(z) \rightarrow \tilde{G}(z) = \tilde{H}(z^N)$$

$$H(\theta) = \tilde{H}(e^{j\theta}) \rightarrow G(\theta) = \tilde{G}(e^{j\theta}) = \tilde{H}(e^{jN\theta})$$
$$= H(N\theta)$$

Example 1



Example 2

$$\widetilde{H}(z) = 1 - z^{-1} = \frac{z-1}{z}$$

Zero: z=1 ; Pole: z=0

$$H(\theta) = 1 - e^{-j\theta} = 2j sin\left(\frac{\theta}{2}\right) e^{-j\frac{\theta}{2}}$$

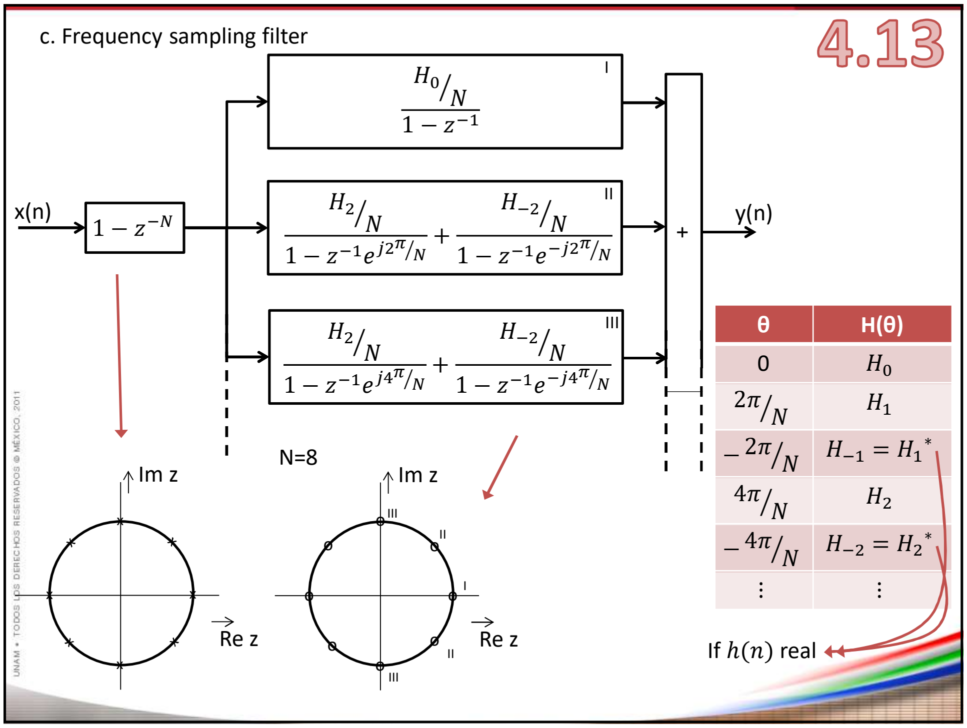
$$\widetilde{H}(z) = 1 - z^{-5} = \frac{z^5-1}{z^5}$$

5 zeros: $z_k = e^{-j2\pi k/5}$

$k = 1,2,3,4$

5 poles in z=0

$$H(\theta) = 1 - e^{-5j\theta} = 2j sin\left(\frac{5\theta}{2}\right) e^{-j\frac{5\theta}{2}}$$



c. Frequency sampling filter

θ	H(θ)
0	H_0
$2\pi/N$	H_1
$-2\pi/N$	$H_{-1} = H_1{}^*$
$4\pi/N$	H_2
$-4\pi/N$	$H_{-2} = H_2{}^*$
⋮	⋮

If $h(n)$ real

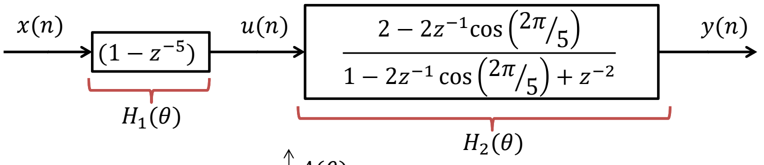
Example: N=5

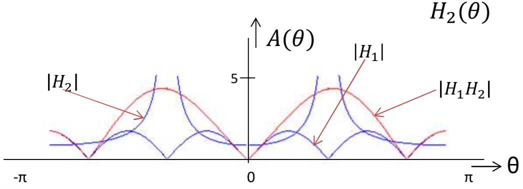
Choose: $H_{-2} = H_0 = H_2 = 0$ and $H_{-1} = H_1 = 5$

Recursive part of $H(z)$:

$$\frac{H_1/_N}{1 - z^{-1}e^{j2\pi/_N}} + \frac{H_{-1}/_N}{1 - z^{-1}e^{-j2\pi/_N}} = \frac{1}{1 - z^{-1}e^{j2\pi/_5}} + \frac{1}{1 - z^{-1}e^{-j\,\pi/_5}}$$

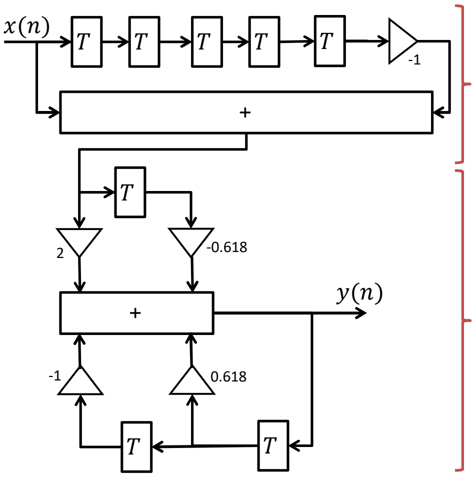
$$= \frac{2 - 2z^{-1}\cos\left(2\pi/_5\right)}{1 - 2z^{-1}\cos\left(2\pi/_5\right) + z^{-2}} \qquad H(z) = (1 - z^{-5})\frac{2 - 2z^{-1}\cos\left(2\pi/_5\right)}{1 - 2z^{-1}\cos\left(2\pi/_5\right) + z^{-2}}$$





Realization

$$H(z) = (1 - z^{-5})\frac{2 - 2z^{-1}\cos\left(2\pi/_5\right)}{1 - 2z^{-1}\cos\left(2\pi/_5\right) + z^{-2}} = (1 - z^{-5})\frac{2 - 0.618z^{-1}}{1 - 0.618z^{-1} + z^{-2}}$$



$(1 - z^{-5})$

$$\frac{2 - 0.618z^{-1}}{1 - 0.618z^{-1} + z^{-2}}$$

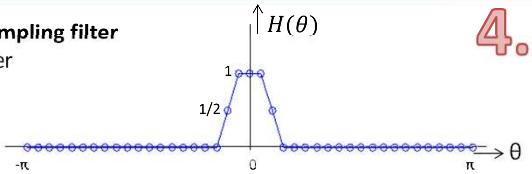
Be careful !

Initial conditions !

Quantization of the coefficients !

Adventages of the frequency sampling filter

Very simple for narrow band filter



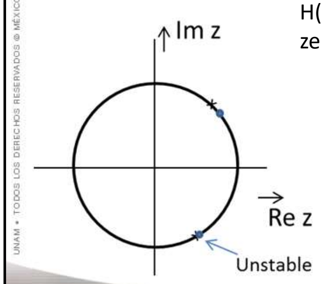
$$H(z) = (1-z^{-N})\left[\frac{{}^{1}/_{N}}{1-z^{-1}} + \frac{{}^{2}/_{N}\left(1-z^{-1}\cos\left({}^{2\pi}/_{N}\right)\right)}{1-2z^{-1}\cos\left({}^{2\pi}/_{N}\right)+z^{-2}} + \frac{{}^{1}/_{N}\left(1-z^{-1}\cos\left({}^{4\pi}/_{N}\right)\right)}{1-2z^{-1}\cos\left({}^{4\pi}/_{N}\right)+z^{-2}}\right]$$

1 multiplication

3 multiplications

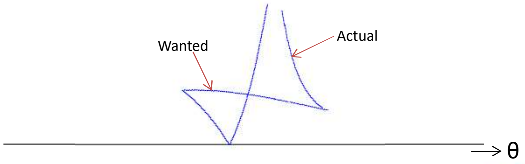
3 multiplications

7 multiplications



Disadvantages:

H(θ) is very sensitive for the actual position of the poles and the zeros (pole/zero cancelation)



d. IIR filters:

For stable system: $\lim_{n\to\infty} h(n) = 0$

Therefore any h(n) can be approximated by h_N(n):

$$h_N(n) = \begin{cases} h(n) & 0 \le n \le N \\ 0 & n > N \end{cases}$$

Therefore any IIR filter can be approximated by a FIR filter of length (N+1)

Not feasible for large N

$$FIR: \quad y(n) = \sum_{i=0}^{N} a_i x(n-i)$$

$$IIR: \quad y(n) = \sum_{i=0}^{M} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

$$\tilde{Y}(z) = \sum_{i=0}^{M} a_i z^{-i}\tilde{X}(z) - \sum_{i=1}^{N} b_i z^{-i}\tilde{Y}(z) = \tilde{X}(z)\sum_{i=0}^{M} a_i z^{-i} - \tilde{Y}(z)\sum_{i=1}^{N} b_i z^{-i}$$

$$\tilde{H}(z) = \frac{\tilde{Y}(z)}{\tilde{X}(z)} = \frac{\sum_{i=0}^{M} a_i z^{-i}}{1+\sum_{i=1}^{N} b_i z^{-i}} = \sum_{k=0}^{N} \frac{A_k}{1-P_k z^{-i}} \qquad (M < N)$$

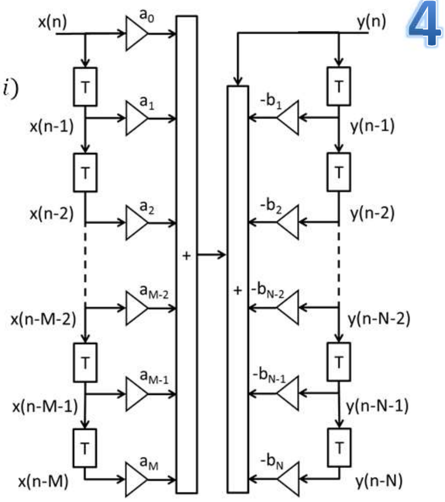
$$h(n) = \sum_{k=0}^{N} A_k {P_k}^n u(n) \qquad (P_k \ real \ or \ complex)$$

Stable IIR filter: $|P_k| < 1 \quad k = 0,1,\ldots,N$

Direct form 1

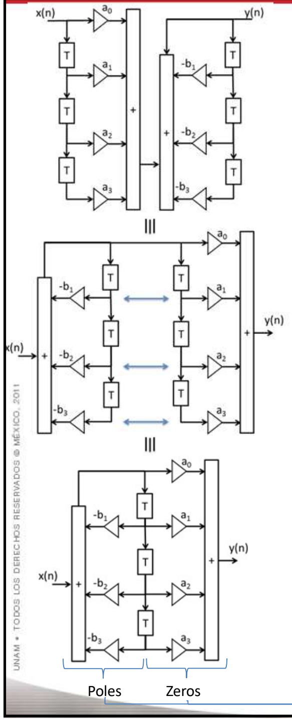
$$y(n) = \sum_{i=0}^{M} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

$$\tilde{H}(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}}$$



$$\tilde{H}(z) = \frac{z^{-M}(a_0 z^M + a_1 z^{M-1} + \cdots + a_M)}{z^{-N}(z^N + b_1 z^{N-1} + \cdots + b_N)} = a_0 z^{N-M} \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-P_1)(z-P_2)\ldots(z-P_N)}$$

- N Poles and M zeros in the z-plane
  N-M zeros in z=0 or M-N poles in z=0
- M+N delay elements
  M_N+1 multiplications and M+N additions for each y(n)



N=M=3

Direct form 1

N+M delay elements

N+M+1 multipliers

Direct form 2

Max(N,M) delay elements

N+M+1 multipliers

"Canonic" $\tilde{H}(z) = \dfrac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$

→ Very sensitive (D.F. 1 and 2)

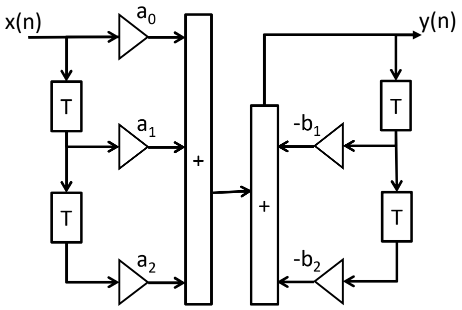
Transposition

4.20

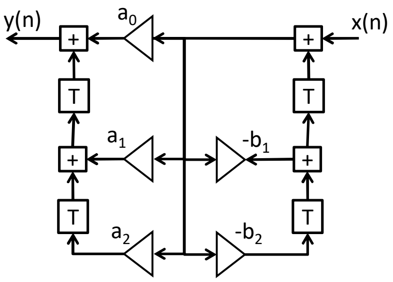
Theorem: If in a digital filter

- Signal flow is reversed (output = input)
- Adders ← Node

The resulting system has the same function as the original structure



Direct form 1



Transposed direct form 1

Cascade structure

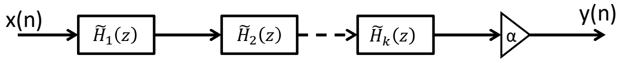
4.21

Assume N=M

$$\tilde{H}(z) = \alpha \frac{\prod_{k=1}^{N}(z - z_k)}{\prod_{k=1}^{N}(z - z_k)} = \alpha \frac{\prod_{k=1}^{N}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - P_k z^{-1})}$$

$$= \alpha \underbrace{\prod_{k=1}^{k_1}\left[\frac{1 - z_k z^{-1}}{1 - P_k z^{-1}}\right]}_{\text{Real poles zeros}} \cdot \underbrace{\prod_{k=1}^{k_2}\left[\frac{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}{1 - b_{1k} z^{-1} - b_{2k} z^{-2}}\right]}_{\text{Complex conjugate poles/zeros}}$$

$$= \alpha \tilde{H}_1(z) \cdot \tilde{H}_2(z) \cdot \cdots \cdot \tilde{H}_k(z) \qquad k = k_1 + 2k_2$$



$\tilde{H}_i(z)$ may be direct form 1 or 2 or…

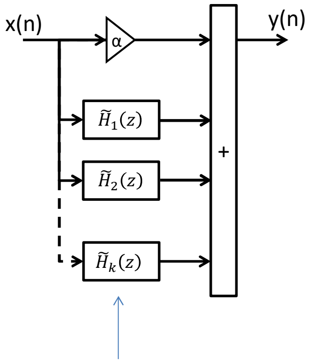
Advantage of this structure

The influence of the finite representation of the coefficients on the transmission function is very small as compared with the direct form (one/two coefficients determine the position of each pole and each zero)

Parallel structure

4.22

$$\widetilde{H}(z) = \alpha \frac{\prod_{k=1}^{N}(1-z_k z^{-1})}{\prod_{k=1}^{N}(1-P_k z^{-1})} = A_0 + \underbrace{\sum_{k=1}^{k_1}\frac{A_k}{1-P_k z^{-1}}}_{\text{Real pole}} + \underbrace{\sum_{k=1}^{k_2}\frac{A_{0k}+A_{1k}z^{-1}}{1-B_{1k}z^{-1}-B_{2k}z^{-2}}}_{\text{Complex conjugate Poles pair}}$$



Direct form 1 or 2 or...

Small influence of coefficients on the position of the poles; this in contrast with the position of the zeros

Digital oscillators

4.23

Some trigonometric formulas:

$$\cos(x \pm y) = \cos(x)\cdot\cos(y) \mp \sin(x)\cdot\sin(y)$$

We write:

$$\cos(\alpha)+\cos(\beta) = cos\left[\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right] + cos\left[\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right]$$

$$= cos\left[\frac{\alpha+\beta}{2}\right]\cdot cos\left[\frac{\alpha-\beta}{2}\right] - \cancel{sin\left[\frac{\alpha+\beta}{2}\right]\cdot sin\left[\frac{\alpha-\beta}{2}\right]}$$

$$+cos\left[\frac{\alpha+\beta}{2}\right]\cdot cos\left[\frac{\alpha-\beta}{2}\right] + \cancel{sin\left[\frac{\alpha+\beta}{2}\right]\cdot sin\left[\frac{\alpha-\beta}{2}\right]}$$

$$= 2cos\left[\frac{\alpha+\beta}{2}\right]\cdot cos\left[\frac{\alpha-\beta}{2}\right]$$

$$\boxed{\cos(\alpha) = 2cos\left[\frac{\alpha+\beta}{2}\right]\cdot cos\left[\frac{\alpha-\beta}{2}\right] - \cos(\beta)}$$

Using this goniometric equations, we are going to design a digital oscillator

4.24

$$\cos(\alpha) = 2cos\left[\frac{\alpha+\beta}{2}\right]\cdot cos\left[\frac{\alpha-\beta}{2}\right] - \cos(\beta)$$

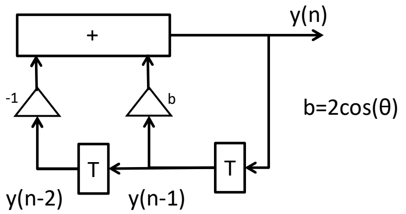
Substitute: $\alpha = n\theta \quad \beta = (n-2)\theta$

Thus: $\frac{\alpha+\beta}{2} = (n-1)\theta \quad and \quad \frac{\alpha-\beta}{2} = \theta$

$$\cos(n\theta) = 2cos[(n-1)\theta]\cdot cos[\theta] - \cos((n-2)\theta)$$

$$\cos(n\theta) = y(n) \quad and \quad 2\cos(\theta) = b$$

$$y(n) = by(n-1) - y(n-2)$$



b=2cos(θ)

The frequency can be tuned by b:

$$b = 2\cos(\theta) \rightarrow \cos(\theta) = {}^{b}/_{2}$$

or:

$$\theta = arc\ cos\left({}^{b}/_{2}\right)$$

Question: $y(n) \overset{?}{=} \cos(n\theta)$

$$\mathrm{y(n)} = \sqrt{\frac{2}{1-\cos(2\theta)}}\, cos\left(n\theta + tg^{-1}\frac{1-\cos(2\theta)}{\sin(2\theta)}\right)$$

If applying δ(n) to start oscillation

4.25

Generation of cos(nθ) ans sin(nθ)

$$\cos(n\theta) = cos[\theta + (n-1)\theta] = \cos(\theta)\cos[(n-1)\theta] - \sin(\theta)\sin[(n-1)\theta]$$

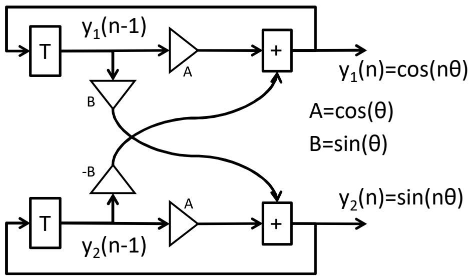
$$\sin(n\theta) = \sin(\theta)\cos[(n-1)\theta] + \cos(\theta)\sin[(n-1)\theta]$$

Substitute: $\cos(n\theta) = y_1(n) \quad \sin(n\theta) = y_2(n)$

$$\cos(\theta) = A \quad and \quad \sin(\theta) = B$$

$$y_1(n) = Ay_1(n-1) - By_2(n-1)$$

$$y_2(n) = By_1(n-1) + Ay_2(n-1)$$



A=cos(θ)

B=sin(θ)

Initial conditions !

Use of tables (ROM)

Example:

$$cos\left(n\frac{\pi}{2}\right)=\begin{cases}1 & n=0+4k\\ 0 & n=1+4k\\ -1 & n=2+4k\\ 0 & n=3+4k\end{cases}\quad k=0,1,2\ldots$$

Store the values 1; 0; -1; 0 in a ROM

In general:

$$cos\left(n\frac{k}{N}2\pi\right)\quad is\ periodic$$

Store N values in a ROM

Remark: If certain symmetry relations exist, less values have to be stored