# 6. Design of digital filters.

## 6.1. The design procedure

The procedure for the design of a digital filter consists of a number of steps that can roughly be described as follows:

step 1: Specification of the requirements. This can be an attenuation scheme, an impulse response, phase characteristic, which may or may not be combined with a set of constraints such as:

- it must be FIR/IIR
- desired number of bits for signals/coefficients
- filter degree or length of impulse response etc.

step 2: Selection of a filtertype (FIR or IIR) if not specified, and of a configuration (direct from I and II, cascade, or parallel or other)

step 3: The approximation. During this step the degree and the filter coefficients are determined that realize the given specifications.

step 4: Determination of the required number of bits for the coefficients so that the specifications are still satisfied. If this appears to be too large, step 3 can be repeated with somewhat increased specifications, or an other filter type or configuration can be selected (step 2).

step 5: Determination of the location and effects of the necessary quantization and overflow non-linearities and their effect on the filter behaviour. From this analysis the required internal wordlength can be determined. In case of a non-satisfactory result in this step it is again possible to recommence in an earlier stage of the procedure.

step 6: The hardware (or software) realization.

In this section we will mainly concentrate on step 3, that is given a specification of the filter requirements, determine the filter degree and/or coefficients.
Section 6.2. will deal with such methods for FIR filters, while section 6.3 deals with IIR filters.

## 6.2. Approximation procedure for FIR filters.

If the filter specifications are given in the time-domain and a desired finite impulse response is given, then the FIR filter design is trivial since a transversal filter with coefficients equal to the nonzero impulse response values will do the job.

We will therefore discuss now some procedures for the case that the desired impulse response is infinite or that the specifications are given in the frequency-domain.

### 6.2.1. Windowing.

Let the desired transmission function $H_d(\theta)$ be specified for $-\pi \leqslant \theta \leqslant \pi$.
The corresponding impulse response $h_d(n)$ will in general be an infinite sequence. The first thing to think of is to approximate $h_d(n)$ by a finite impulse response $h_N(n)$ with length N such that:

$$h_N(n) = \begin{cases} h_d(n) & o \leqslant n \leqslant N-1 \\ 0 & \text{elsewhere} \end{cases} \tag{6.1}$$

Introducing the function

$$p_N(n) = \begin{cases} 1 & o \leqslant n \leqslant N-1 \\ 0 & \text{elsewhere} \end{cases} \tag{6.2}$$

eq.(6.1) can be written as:

$$h_N(n) = h_d(n) . \; p_N(n). \tag{6.3}$$

Eq.(6.3) is a special form of windowing, which is the multiplication of an infinite time series with a finite duration "window function" $w_N(n)$.

$$h_N(n) = h_d(n) . \; w_N(n)$$

Due to its particular form $p_N(n)$ is called a <u>rectangular window</u>.
Using the properties of the FTD, the frequency domain interpretation of this windowing technique can be derived:

$$H_N(\theta) = H_d(\theta) * W_N(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\xi) \, W_N(\theta - \xi) \, d\xi . \tag{6.4}$$

where

$$W_N(\theta) = \sum_{n=o}^{N-1} w_N(n) \, e^{-jn\theta}. \tag{6.5}$$

For the rectangular window the spectrum is given by:

$$P_N(\theta) = \frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}} . \, e^{-j\frac{N-1}{2}\theta} \tag{6.6}$$

The modulus of this function is schematically shown in fig.6.1. for N = 24.

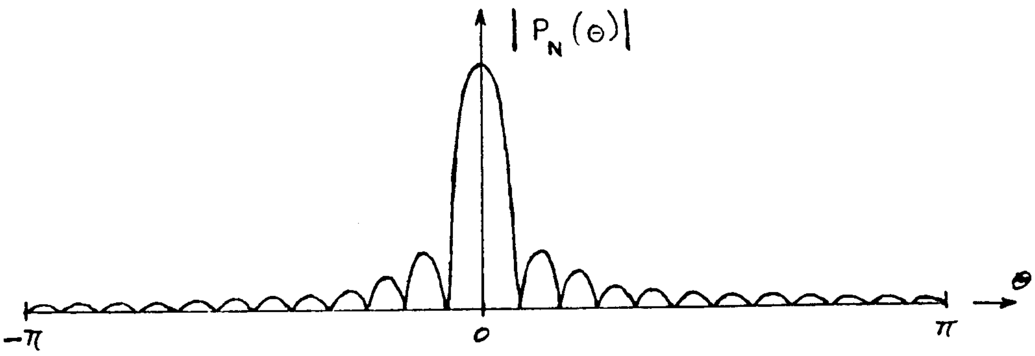



fig.6.1.

To determine the influence of this window function on the transmission function we must solve the convolution equation (6.4). As an example consider the case of a lowpass filter, and for simplicity assume that both the desired filter and the realizable filter are zero-phase filters, i.e. their impulse response is symmetrical around n=0. In that case the window function will likewise be made symmetrical around n=0: $w_N(n) = w_N(-n)$.

(The filter so obtained will be non causal, but can be made causal by a mere shift of the impulse response).

Thus let

$$H_d(\theta) = \begin{cases} 1 & |\theta| < \theta_c \\ 0 & \theta_c < |\theta| \leqslant \pi \end{cases}$$

Taking a rectangular window, which when symmetrical around n=0 must have N odd, gives the spectrum:

$$W_N(\theta) = \frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}}$$

The resulting low-pass filter has a transmission function as shown in fig. 6.2.

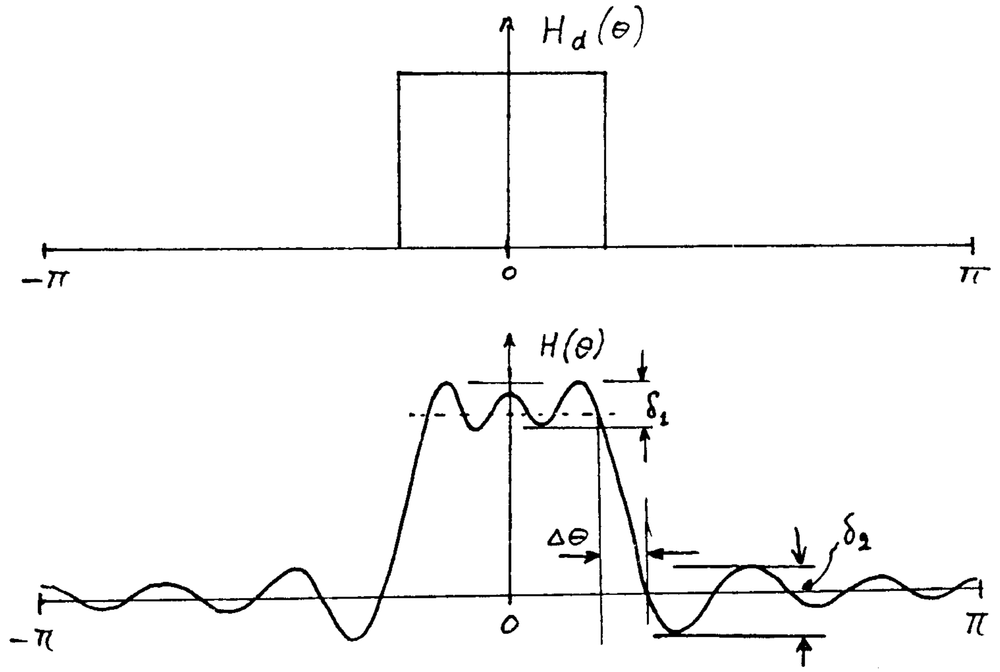



fig.6.2.

Due to the truncation of the impulse response caused by the rectangular window the following effects occur:

1) ripples in the passband and in the stopband caused by the sidelobes of the spectrum of $P_N(\theta)$

2) a transition region between passband and stopband of width $\Delta\theta$, determined by the width of the main lobe of $W_N(\theta)$.

With a rectangular window the ripples in passband and stopband will not decrease in amplitude when N is increased but the width of both the main lobe and the side lobes decrease.

In general what we want is that both the ripples and $\Delta\theta$ can be made small by taking a sufficiently large N. But then a different window function must be taken, i.e. a function that more "smoothly" goes to zero near the end points. Several of these window functions have been proposed of which a number are listed in table 6.1. All these functions have the property that their main lobe is wider than for the rectangular window (resulting in an increase of the transition bandwidth) but the amplitude of the sidelobes is less (giving less passband ripple and larger stopband attenuation). More details concerning these window functions can be found in all modern text books on digital signal processing.

Table 6.1. Window function.

| name | $w_N(n)$ | amplitude side lobe | width of main lobe |
|---|---|---|---|
| rectangular | $1 \qquad 0 \leq n \leq N-1$ | -13dB | $4\pi/N$ |
| Bartlett | $\begin{cases} 2n/(N-1) & 0 \leq n \leq \frac{N-1}{2} \\ 2 - 2n/(N-1) & \frac{N-1}{2} \leq n \leq N-1 \end{cases}$ | -25dB | $8\pi/N$ |
| Hanning | $\frac{1}{2}\left[1 - \cos(\frac{2\pi n}{N-1})\right] \quad 0 \leq n \leq N-1$ | -31dB | $8\pi/N$ |
| Hamming | $0.54 - 0.46 \cos(\frac{2\pi n}{N-1}) \quad 0 \leq n \leq N-1$ | -41dB | $8\pi/N$ |
| Blackman | $0.42 - 0.5 \cos(\frac{2\pi n}{N-1}) + 0.08 \cos(\frac{4\pi n}{N-1}) \quad 0 \leq n \leq N-1$ | -57dB | $12\pi/N$ |
| Kaiser | $\frac{I_o(\beta\sqrt{1-(\frac{2n}{N-1}-1)^2})}{I_o(\beta)} \quad 0 \leq n \leq N-1$ | -30 ∿ -100dB depends on $\beta$ | > $4\pi/N$ depends on $\beta$ |

In conclusion, windowing techniques are relatively simple, and form a viable design method for filters for which a piecewise constant transmission function is desired.

### 6.2.2. Frequency sampling.

In applications such as data transmission, filter specifications are often not in terms of a passband with a prescribed ripple, a stopband with prescribed attenuation and a transition band with prescribed width. Rather what one wants is a smooth transmission function satisfying certain symmetry relations around some frequency to fulfil one of the Nyquist criteria. In that case a windowing technique is not easily applicable.

A method suitable for this case is frequency sampling. Let $H_d(\theta)$ be specified. Then on the fundamental interval $(0,2\pi)$ N equidistant points are considered.

Let

$$\theta_k = \frac{2\pi}{N} k \qquad k=o,\ldots,N-1 \qquad (6.7)$$

and define the numbers

$$\bar{H}_k \stackrel{\wedge}{=} H_d(\theta_k) \qquad k=o,\ldots,N-1 \qquad (6.8)$$

These numbers satisfy the relation:

$$\bar{H}_k = \bar{H}^*_{N-k} \qquad k=1,---, N-1 \qquad (6.9)$$

Therefore, when the inverse discrete Fourier tranform (IDFT , see Signal Analysis, chapter 8) is applied to these numbers we get N real valued numbers $\bar{h}_n$:

$$\bar{h}_n = \frac{1}{N} \sum_{k=0}^{N-1} \bar{H}_k \, e^{j\frac{2\pi}{N}kn} \qquad n=0,---,N-1 \qquad (6.10)$$

Now we take an FIR filter with impulse response

$$h_N(n) = \begin{cases} \bar{h}_n & 0 \leqslant n \leqslant N-1 \\ 0 & \text{elsewhere} \end{cases} \qquad (6.1$$

The corrsponding transmission function is

$$H_N(\theta) = \sum_{n=-\infty}^{\infty} h_N(n) \, e^{-j\theta n}$$

$$= \sum_{n=0}^{N-1} \bar{h}_n \, e^{-j\theta n}$$

and thus

$$H_N(\theta_k) = \sum_{n=0}^{N-1} \bar{h}_n \, e^{-j\theta_k n} = \sum_{n=0}^{N-1} \bar{h}_n e^{-j\frac{2\pi}{N} kn} \qquad (6.$$

The last expression in (6.12) is the DFT equation and thus we can conclude that

$$H_N(\theta_k) = \bar{H}_k = H_d(\theta_k) \qquad (6.13)$$

Therefore the filter with impulse response specified by eq.(6.11) has a transmission function that is equal to the desired frequency response on N equidistant points of the fundamental interval of the spectrum. In between these points the two transmission functions will be different in general.
$H_N(\theta)$ can after some manipulation be written as

$$H_N(\theta) = \frac{1}{N} \sum_{k=0}^{N-1} \bar{H}_k \, e^{-j(\theta-\theta_k)(N-1)/2} \; \frac{\sin N\,(\theta - \theta_k)/2}{\sin\,(\theta - \theta_k)/2} \qquad (6.14)$$

which means that it is an interpolated version of the discrete set $\{\bar{H}_k\}$ with interpolation function $e^{-j\theta(N-1)/2}\sin(N\,\theta/2)/\sin(\theta/2)$.
Especially in the neighbourhood of discontinuities in $H_d(\theta)$ large ripples can be observed in $H_N(\theta)$, but for a smooth $H_d(\theta)$ very nice results can be obtained.

For filters with a transition region it is possible to obtain better results by taking the values of $\bar{H}_k$ for $\theta_k$ in the transition region as free parameters and applying optimization techniques. We will not consider these techniques here, however.

### 6.2.3. Optimum linear phase FIR filters.

As was remarked before linear phase filters have an impulse response that satisfies the symmetry relation of eq.(4.12).Four different cases can be distinguished.

<u>case 1</u>: N odd, positive symmetry: $h(N-1-n) = h(n)$
Writing $N=2K+1$ the symmetry relation can be rewitten as:

$$h(K-n) = h(K+n) \qquad n=1,\ldots\ldots,K$$

From this it follows that

$$H(\theta) = e^{-jK\theta}\cdot\sum_{n=0}^{K} b_1(n)\cos n\theta \tag{6.15}$$

where $b_1(n) = \begin{cases} 2h(K-n) & n=1,\ldots,K. \\ h(K) & n=0. \end{cases}$

<u>case 2</u>: N even, $(N=2K+2)$
positive symmetry: $h(N-1-n) = h(n)$

A rather involved computation reveals that in this case the transmission function can be written as:

$$H_2(\theta) = e^{-j\frac{N-1}{2}\theta}\cos\frac{\theta}{2}\cdot\sum_{n=0}^{K} b_2(n)\cos n\theta \tag{6.16}$$

where the coefficients $b_2(n)$ are related to the original coefficients $h(n)$ in a rather complicated way.

<u>case 3</u>: N odd, $(N = 2K+3)$
negative symmetry: $h(N-1-n) = -h(n)$

Similarly as in case 2 the transmission function can be brought into the form:

$$H_3(\theta) = e^{-j\frac{N-1}{2}\theta}\cdot\sin\theta\sum_{n=0}^{K} b_3(n)\cos n\theta \tag{6.17}$$

<u>case 4</u>: N even (N = 2K+2)
negative symmetry: h(N-1-n) = - h(n)

In this case

$$H_4(\theta) = e^{-j\frac{N-1}{2}\theta} \cdot \sin\frac{\theta}{2} \cdot \sum_{n=0}^{K} b_4(n)\cos n\theta \qquad (6.18)$$

Therefore all linear phase FIR filters have a transmission function of the form:

$$H(\theta) = e^{-j\frac{N-1}{2}\theta} \cdot Q(\theta) \cdot H_o(\theta) \qquad (6.19)$$

where

$$Q(\theta) = \begin{cases} 1 & \text{case 1} \\ \cos\theta/2 & \text{case 2} \\ \sin\theta & \text{case 3} \\ \sin\theta/2 & \text{case 4} \end{cases} \qquad (6.20)$$

and $H_o(\theta)$ is a function of the form.

$$H_o(\theta) = \sum_{n=0}^{K} b(n)\cos n\theta \qquad (6.21)$$

From (6.20) it can be concluded that for case 2 $H(\pi)=0$ so that high-pass filters cannot be realized.
For case 3 $H(o) = H(\pi) = 0$, which means that only filters with a bandpass characteristic are possible. In case 4 $H(o) = 0$ and thus low-pass filters cannot be made with an even length impulse response with negative symmetry.
Ignoring the phase factor $e^{-j\frac{N-1}{2}\theta}$ the difference between the desired and the actual tramsission function is

$$H_d(\theta) - Q(\theta) \cdot H_o(\theta) \qquad (6.22)$$

Now, using methods of numerical mathematics it is possible to determine the coefficents b(n)such that the weighted error

$$E(\theta) = W(\theta)\left[H_d(\theta) - Q(\theta)H_o(\theta)\right] \qquad (6.23)$$

is minimized in some sense, where $W(\theta)$is any appropriate weighting function.

Let $\Theta$ be the interval on which the minimization is required. (union of passbands and stopbands). An optimum solution is obtained if the minimization of E is performed in the Chebychev sense, i.e.if the b(n) are found so as to minimize the maximum absolute value of $E(\theta)$:

$$\min_{\{b(o),\dots,b(k)\}} \left[ \max_{\theta\in\Theta} |E(\theta)| \right] \tag{6.24}$$

The solution so obtained is optimum in the sense that no other filter has a smaller peak error over the entire interval of optimization $\Theta$. The transmission function will be equiripple which means that all passband ripples will have the same amplitude $\delta_1$ and all stopband ripples the value $\delta_2$ for a two-band design (a similar statement holds for a multiband design).

It will not be necessary to go into details concerning the numerical methods of the Chebychev approximation. A number of sophisticated computer programs exist that can compute the filter coefficients for given specifications of low-pass, bandpass, high-pass, bandstop and some other types of filters. A computer program developed at Rice university by Parks and McClellan has become quite popular. The Fortran text of this program can be found in the book by Rabiner and Gold. pp.187-204.

Whatever program will be used, it will always be necessary to have an initial guess of the filter length N that is required for satisfying the filter specifications. A too small N will make that these specifications cannot be satisfied, too large an N results in a filter which is more complex than necessary. Therefore the optimization has often to be performed a number of times with different values of N to find the one which gives the "best performance". A good initial guess may considerably reduce the computational expenditure of this procedure. To this end Rabiner et al. have determined experimentally (i.e. by computing a very large number of different filters) a set of design formula, among which one that gives an estimate for the filter length N required for a low-pass filter with passband ripple $\pm\,\delta_1$, stopband attenuation $\delta_2$ and transition bandwidth $\Delta F$:

$$\hat{N} = \frac{D(\delta_1,\delta_2)}{\Delta F} - f(\delta_1,\delta_2).\Delta F + 1 \tag{6.25}$$

where

$$D(\delta_1,\delta_2)= \left[5.309.10^{-3}(\log \delta_1)^2 + 7.114.10^{-2} \log \delta_1 - 4.761.10^{-1}\right].\log\delta_2 - 2.66\ 10^{-3}\ (\log\delta_1)^2 - 5.941.10^{-1} \log \delta_1 - 4.278.10^{-1} \tag{6.26}$$

and

$$f(\delta_1,\delta_2)= 11.01217 + 0.51244\ (\log \delta_1 - \log \delta_2) \tag{6.27}$$

## 6.3. Design procedures for IIR filters.

The design procedures that have been developed for IIR filters can roughly be divided into two groups:

1. Optimization methods in the digital frequency domain. In these methods the problem of finding the filter coefficients is treated as an optimization problem that is solved by applying well-known optimization algorithms as for example linear programming. The most difficult part of the procedure is to bring the problems into the format required for such an algorithm.
   These methods require knowledge of optimization theory, and therefore we will not discuss them here,

2. Methods that transform the design of a digital filter into a design of an analog filter by linking in some way or the other the digital and analog worlds. The analog design is a well-solved problem, since many algorithms, design charts, tables etc. exist for designing all types of analog filters. The basic difference in these techniques is the way in which the analog and digital worlds are tied together. This latter problem is very interesting, and its study may lead to more insight into the similarities and differences between analog and digital systems.

### 6.3.1. Relation between analog and digital systems.

Consider the analog system in fig.6.3, consisting of a resistance R and a capacitance C.

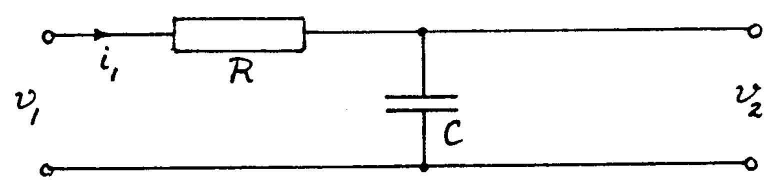



fig.6.3.

If $v_1(t)$ is viewed as the input signal and $v_2(t)$ as the output signal the following ways exist to describe the transmission of this circuit.

a) The differential equation:

$$\tau \frac{dv_2(t)}{dt} + v_2(t) = v_1(t) , \qquad \tau = R.C \tag{6.28}$$

b) The impulse response:

$$v_2(t) = h_a(t) * v_1(t) \tag{6.29}$$

where
$$h_a(t) = \frac{1}{\tau} e^{-t/\tau} . u_a(t) \tag{6.30}$$

c) The transmission function $H_a(\omega)$ which is the Fourier transform of $h_a(t)$:

$$H_a(\omega) = \frac{1}{1+j\omega\tau} \tag{6.31}$$

d) The system function $\tilde{H}_a(p)$ which is the Laplace transform of $h_a(t)$:

$$\tilde{H}_a(p) = \frac{1}{1+p\tau} \tag{6.32}$$

This latter function is fully characterised by the place of the pole

$$p_o = -\frac{1}{\tau} \tag{6.33}$$

A similar set of descriptions exists for the simple digital system of fig.6.4.

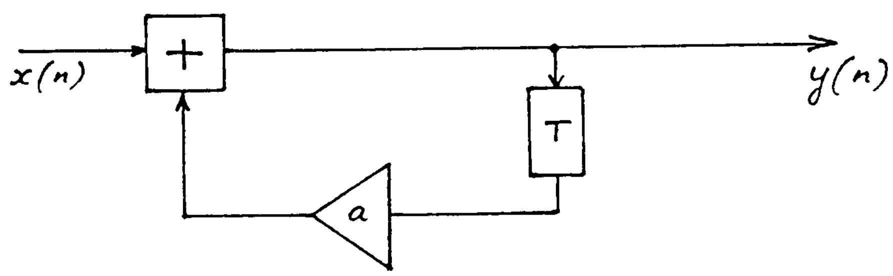



fig.6.4.

a) The difference equation:

$$y(n) = a\ y(n-1) + x(n) \tag{6.34}$$

b) The impulse response:

$$y(n) = h(n) * x(n) \tag{6.35}$$

where

$$h(n) = a^n\ u(n) \tag{6.36}$$

c) The transmission function:

$$H(\theta) = \frac{1}{1 - a.e^{-j\theta}} \tag{6.37}$$

d) The system function:

$$\tilde{H}(z) = \frac{1}{1 - a\,z^{-1}} \tag{6.38}$$

or its pole

$$z_o = a \tag{6.39}$$

Starting from any of the descriptions of the analog circuit a corresponding digital circuit can be determined which in some sense resembles the analog system. Several of such techniques will now briefly be reviewed, not as much because each of them provides an efficient method for designing IIR filters, but rather because they yield more insight into the relation and differences between analog and digital systems.

## 6.3.2. Mapping of differential equations.

The differential equations of the analog filter can be mapped into a set of difference equations by the transformation:

$$y_a(t) \Rightarrow y(n) \tag{6.40}$$

$$\frac{d}{dt}\, y_a(t) \Rightarrow \frac{y(n) - y(n-1)}{T}$$

where T in principle may be any normalization constant, but in practice will be taken equal to the sampling period.

If $y(n)$ is obtained from $y_a(t)$ by sampling with rate $^1/T$, then it will be clear that for $T \to 0$ the operation

$$\frac{y(n) - y(n-1)}{T} = \frac{y_a(nT) - y_a(nT-T)}{T}$$

will closely resemble the differentiation operation.
We therefore may expect that for low frequencies the digital system so obtained will behave similar as the analog prototype.
Returning to the previous example we get from eq. (6.28) with $v_1(t) \Rightarrow x(n)$, $v_2(t) \Rightarrow y(n)$:

$$y(n) . \left(\frac{\tau}{T} + 1\right) - \frac{\tau}{T}\, y(n-1) = x(n)$$

or with $a = \tau/(T+\tau)$, $b = {}^T/(T+\tau)$

$$y(n) = a\, y(n-1) + b\, x(n)$$

thus

$$h(n) = a^n . b\, u(n),$$

$$H(\theta) = \frac{b}{1 - ae^{-j\theta}},$$

$$\tilde{H}(z) = \frac{b}{1 - a\, z^{-1}},$$

Only for $T/\tau \ll 1$ does $a^n b$ approximate $e^{-nT/\tau}$ for small values of n, which means that the impulse response of the digital system will in general differ considerably from $h_a(nT)$, i.e. from a sampled version of $h_a(t)$.

As was already hypothesized, for small values of $\theta$ the transmission functions will not be very different.
Indeed for $\theta \gtrsim o$ we have

$$H(\theta) \approx \frac{b}{1 - a\,(1-j\theta)} = \frac{1}{1 + j\theta\,\frac{\tau}{T}}$$

which means

$$H(\omega T) \approx \frac{1}{1 + j\omega\tau} = H_a(\omega) \qquad \omega \to 0$$

Finally the relation between $\tilde{H}_a(p)$ and $\tilde{H}(z)$ can be derived from the transformation given by eq. (6.40).

$$\begin{array}{ccc} \frac{d}{dt}\, y_a(t) & \Longrightarrow & \frac{1}{T}\,\left[\, y(n) - y(n-1) \right] \\ \mathcal{L}\downarrow & & \downarrow \mathcal{Z} \\ p.\tilde{Y}_a(p) & \Longrightarrow & \frac{1}{T}\,(1-z^{-1})\,\tilde{Y}(z) \end{array} \qquad (6.41)$$

The interpretation of (6.41) is that (6.40) specifies a relation between p-domain and z-domain of the form:

$$p = \frac{1}{T}\,(1 - z^{-1}) \qquad (6.42)$$

or

$$z = \frac{1}{1 - pT} \qquad (6.43)$$

Therefore

$$\tilde{H}(z) = \tilde{H}_a\left( \frac{1}{T}(1 - z^{-1}) \right) \qquad (6.44)$$

To see the effect of the mapping (6.43) we can determine the mapping of the imaginary axis $p = j\omega$, which is given by:

$$z = \frac{1}{1 - j\omega T} \quad , \quad -\infty < \omega < \infty$$

The locus of this mapping is a circle with radius 1/2 in the z-plane with centre at z = 1/2, see fig.6.5.

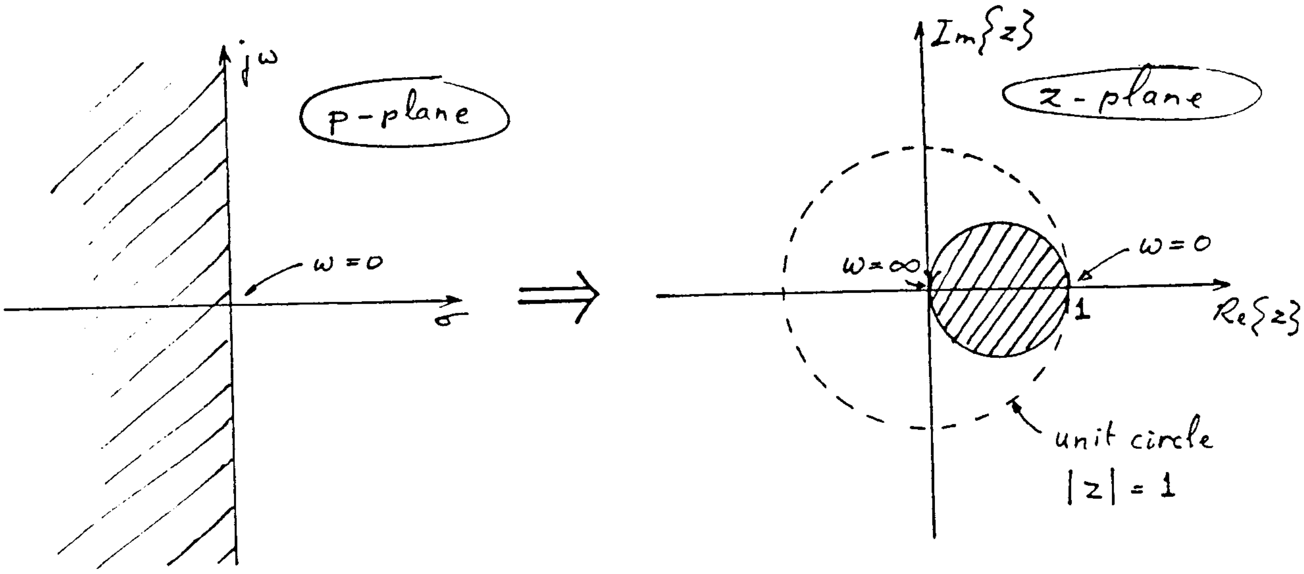



fig.6.5.

It can be shown that the mapping (6.43) maps the left half of the p-plane into the interior of this circle. Therefore, applying the mapping to a stable analog filter results in a stable digital filter, but a digital filter so obtained can only have poles in a small part of the region inside the unit circle, and thus this technique is certainly not suitable for designing general types of digital filters, and certainly not high-pass filters.

The transmission function $H(\theta)$ of the digital filter is equal to the system functions when evaluated on the unit cirle $|z| = 1$. For the analog filter the system function must be evaluated on the imaginary axis to obtain the transmission function.

From fig.6.5. it can immediately be seen that only for ω=o the identity

$$H(\omega T) = \tilde{H}(e^{j\omega T}) = \tilde{H}_a(j\omega) = H_a(\omega)$$

holds, so that only for $\omega \approx 0$ we have that $H(\omega T) \approx H_a(\omega)$ as we had derived before for the special example.

The choice of the mapping in (6.40) was rather arbitrary, and we could just as well have taken the forward difference

$$\frac{d}{dt}\, y_a(t) \Longrightarrow \frac{1}{T}\, \left[y(n+1) - y(n)\right] \tag{6.45}$$

resulting in the p-plane to z-plane mapping

$$p = \frac{1}{T}\,(z-1) \tag{6.46}$$

$$z = pT + 1 \tag{6.47}$$

Intuitively one could have the impression that there should be little difference between the two mappings defined by (6.40) and (6.45). This is not so.
If we display the mapping given by (6.47) we see that the $j\omega$-axis in the p-plane maps to the straight line $z = j\omega T+1$ and the left half of the p-plane is mapped into the half plane to the left of this line, as shown in fig. 6.6.

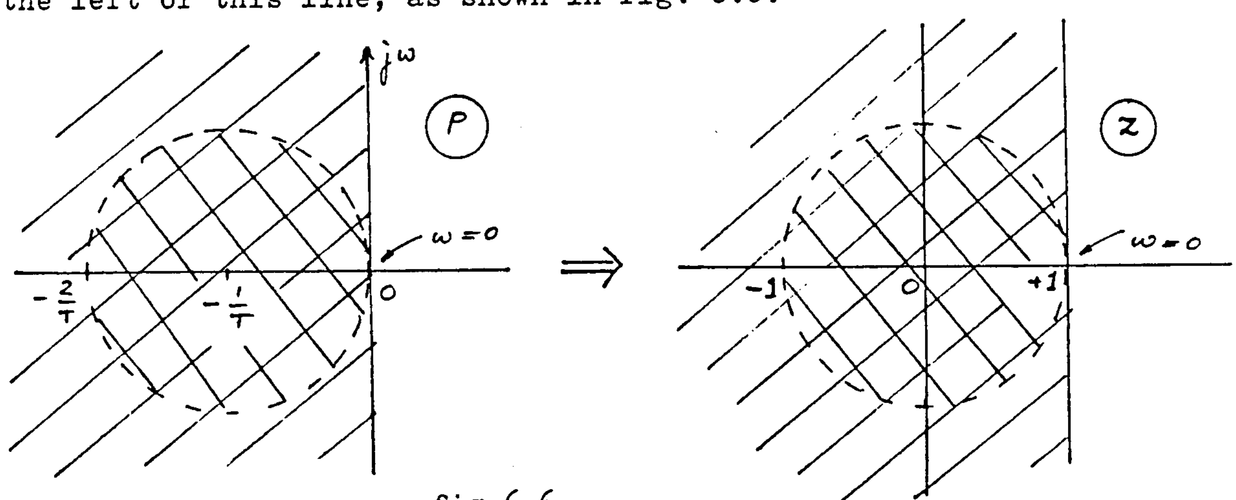



fig.6.6.

As before we see that only for $\omega \approx 0$ we have that

$$H(\omega T) = \tilde{H}(e^{j\omega T}) \approx \tilde{H}_a(j\omega) = H_a(\omega)$$

But, using this mapping it cannot be guaranteed that a stable analog filter results in a stable digital filter. Only if the analog filter had all its poles inside the cross-hatched circular region will the digital filter be stable too.

Many more mappings of the differentials like (6.40) and (6.45) can be defined each with its own properties but none of these really provides an attractive method for designing digital filters.

### 6.3.3. Impulse invariant transformation.

The previous technique had the property that the impulse response of the digital filter in general did not much resemble that of the analog prototype. This means that if we desire some specific properties for the impulse response, for example, little overshoot, equidistant zero values or whatever, and we have an analog filter which has these properties, then the filter resulting from applying the previous technique will in general not have these properties. In cases where it is important to maintain certain characteristics of the impulse response the impulse invariant transformation can be used. In that case what one does is to take

$$h(n) = h_a(nT) \tag{6.48}$$

which means that the impulse response of the D.F. is a sampled version of the impulse response of the analog prototype filter.
Two questions now arise:
1) can a filter with this impulse response always be realized ?

2) what is the corresponding transmission function? The answer to the first question is positive if the analog prototype filter has a rational system function in p, and thus can be written in the form

$$\tilde{H}_a(p) = \sum_{i=1}^{N} \frac{a_i}{p - p_i} \tag{6.49}$$

This means that

$$h_a(t) = \sum_{i=1}^{N} a_i\, e^{p_i t}\, u_a(t) \tag{6.50}$$

and thus

$$h(n) = \sum_{i=1}^{N} a_i (e^{p_i T})^n\, u(n) \tag{6.51}$$

Therefore

$$\tilde{H}(z) = \sum_{i=1}^{N} \frac{a_i}{1 - e^{p_i T} z^{-1}} \tag{6.52}$$

If $\tilde{H}_a(p)$ has real coefficients then either p is real (and negative if $H_a$ is stable) and then $a_i$ is real or there exist two complex conjugate poles $p_i$ and $p_j$ such that

$$p_i = p_j^* = \sigma_i + j\omega_i , \quad a_i = a_j^* = \alpha_i e^{j\gamma_i}$$

Applying this to eq. (6.52) gives:

$$\tilde{H}(z) = \sum_{\substack{\text{real} \\ \text{poles}}} \frac{a_i}{1-e^{p_i T} z^{-1}} + \sum_{\substack{\text{complex} \\ \text{poles}}} \frac{2\alpha_i \cos\gamma_i - 2z^{-1}\alpha_i \rho_i \cos(\gamma_i - \omega_i T)}{1-(2\rho_i \cos\omega_i T)z^{-1} + \rho_i^2 z^{-2}} \quad (6.5$$

$$\rho_i = e^{\sigma_i T}$$

and this system function can be composed of first and second order sections with real coefficients. This proves that indeed a digital filter exists that has the given impulse response.

To obtain the corresponding transmission function we could use (6.53) and evaluate this expression on the unit circle. It is easier, however, to use the fact that h(n) is obtained by sampling $h_a$(t Using eq.(2.7) we find immediately:

$$H(\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a \left( \frac{\theta + k.2\pi}{T} \right) \quad (6.5$$

Therefore $T.H(\omega T)$ will only be equal to $H_a(\omega)$ if this latter function is bandlimited to $\pi/T$, but this will never be exactly so because we required that the system function has the form of eq. (6.49). Therefore always aliasing will occur.

For the example of fig.6.3 this technique gives

$$\tilde{H}_a(p) = \frac{1/\tau}{p + 1/\tau} \Rightarrow \tilde{H}(z) = \frac{1/\tau}{1 - e^{-T/\tau} z^{-1}}$$

and thus

$$H_a(\omega) = \frac{1}{1+j\omega\tau} \Rightarrow H(\omega T) = \frac{1/\tau}{1-e^{-T/\tau} e^{-j\omega T}}$$

Fig.(6.7) shows the two functions $H_a(\omega)$ and $H(\omega T)$. T on a logarithmic scale.

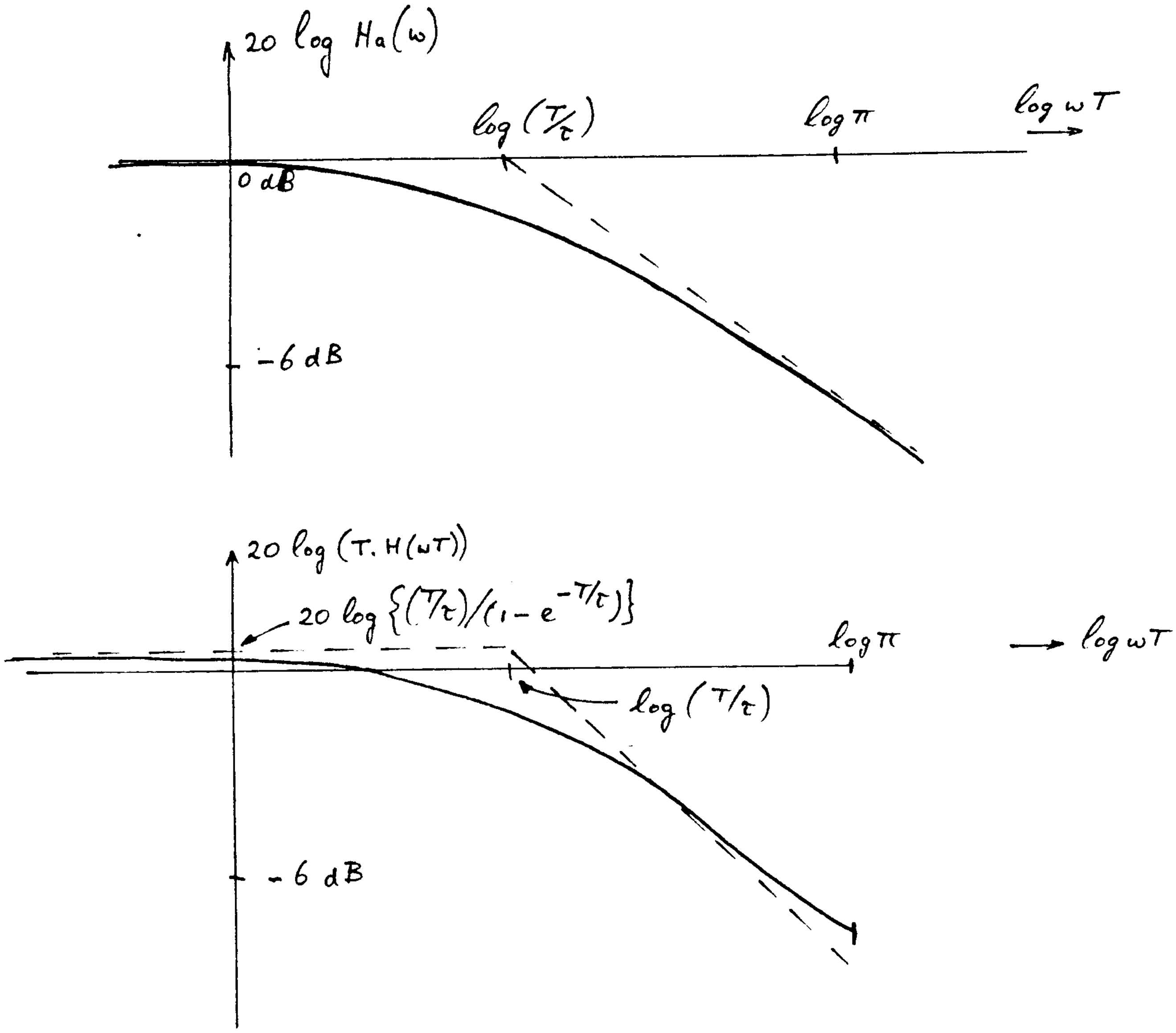



fig.6.7.

In conclusion, the impulse invariant transformation is useful in a design where constraints are imposed on the impulse response. With this technique poles of the analog prototype filter at $p=p_i = \sigma_i+j\omega_i$ in the p-domain are mapped into poles at $z=z_i= e^{(\sigma_i+j\omega_i)T}$ in the z-domain, but a similar mapping does not exist for the zeros. This mapping of the poles guarantees that if the prototype filter is stable ($\sigma_i < 0$ $i=1,---,N$) then the digital filter will be stable too ( $|z_i| = e^{\sigma_i T} < 1$).
In general the transmission function of the digital filter will differ from that of the analog prototype filter especially for $\theta \approx \pi$, due to aliasing.

## 6.3.4. The bilinear transformation.

Instead of starting with the time domain descriptions such as differential equations or impulse responses we can also start with the frequency domain specifications. What we want is a digital filter with system function $\tilde{H}(z)$, such that for $|z| = 1$ i.e. on the unit circle of the z-plane.

$$\tilde{H}(z)\Big|_{z=e^{j\theta}} = H(\theta)$$

satisfies a set of specifications. On the other hand, the analog filter from which we want to derive the D.F. is characterised by $\tilde{H}_a(p)$ in the p-domain and its transmission function can be found by evaluating $\tilde{H}_a(p)$ on the imaginary axis:

$$\tilde{H}_a(p)\Big|_{p=j\omega} = H_a(\omega)$$

It is therefore quite natural to look for a mapping of the p-plane to the z-plane that has the following properties:

1) The mapping must be a rational function (to ensure realizability)
2) The imaginary axis in the p-plane must map onto the unit circle (to connect $H_a(\omega)$ with $H(\theta)$) and such that
3) the left half of the p-plane maps into the inside of the unit circle (to maintain stability)
4) p=o maps into z=1 (to maintain low pass characteristics $\omega=o \Rightarrow \theta=o$)
5) $p=j\infty$ maps into z=-1 (to maintain high pass characteristics $\omega=\infty, \theta=\pi$).

A simple transformation that satisfies all these requirements is:

$$z = \frac{\Omega_o+p}{\Omega_o-p} \qquad (6.55)$$

where $\Omega_o$ is a suitable (positive) normalization factor, to which we will return later.
The inverse transform is given by

$$p = \Omega_o \cdot \frac{z-1}{z+1} \qquad (6.56)$$

Both equations (6.55) and (6.56) are bilinear forms which explains the name of the method.
It is easily verified that conditions 1 through 5 are satisfied.
To see how the $j\omega$-axis maps into the unit circle we set $p=j\omega$, $z=e^{j\theta}$ and insert this into (6.56):

$$j\omega = \Omega_o \cdot \frac{e^{j\theta}-1}{e^{j\theta}+1} = \Omega_o \cdot j \cdot \tan\frac{\theta}{2}$$

Thus

$$\omega = \Omega_o \tan \frac{\theta}{2}. \qquad (6.57)$$

$$\theta = 2 \arctan \frac{\omega}{\Omega_o}. \qquad (6.58)$$

Eq.(6.58) shows that the whole frequency axis $(-\infty,\infty)$ of $\omega$ is mapped onto a fundamental interval $(-\pi,\pi)$ of the $\theta$-axis.

How do we use this bilinear transformation ?
The best way to demonstrate this is by an example.
Assume that a lowpass digital filter is required satisfying the following specifications (see fig.6.8)

$$\begin{aligned} 1-\delta_1 < |H(\theta)| < 1 \qquad & |\theta| < \theta_p \\ |H(\theta)| < \delta_2 \qquad & \theta_s < |\theta| < \pi \end{aligned} \qquad (6.59)$$

Step 1. The bilinear transform specifies a mapping of the unit circle onto the $j\omega$-axis given by eq. (6.57). This (inverse) transform can be used to transform the specifications of eq. (6.59) to a corresponding set of specifications for an analog filter.
For this example we have:

$$\begin{aligned} 1-\delta_1 < |H_a(\omega)| < 1 \qquad & |\omega| < \omega_p \\ |H_a(\omega)| < \delta_2 \qquad & \omega_s < |\omega| < \infty \end{aligned} \qquad (6.60)$$

where

$$\begin{aligned} \omega_p &= \Omega_o \tan \frac{\theta_p}{2} \\ \omega_s &= \Omega_o \tan \frac{\theta_s}{2} \end{aligned} \qquad (6.61)$$

This is shown in fig. 6.8.

Step 2. An analog filter is sought that satisfies the specifications (6.60). To this end any analog filter approximation procedure may be used. A very convenient way is to use a book with filter tables, and if possible, one that specifies the pole and zero-locations, such as for example

E.Christian, E.Eisenmann
Filter design tables and graphs
J.Wiley, New York 1966.

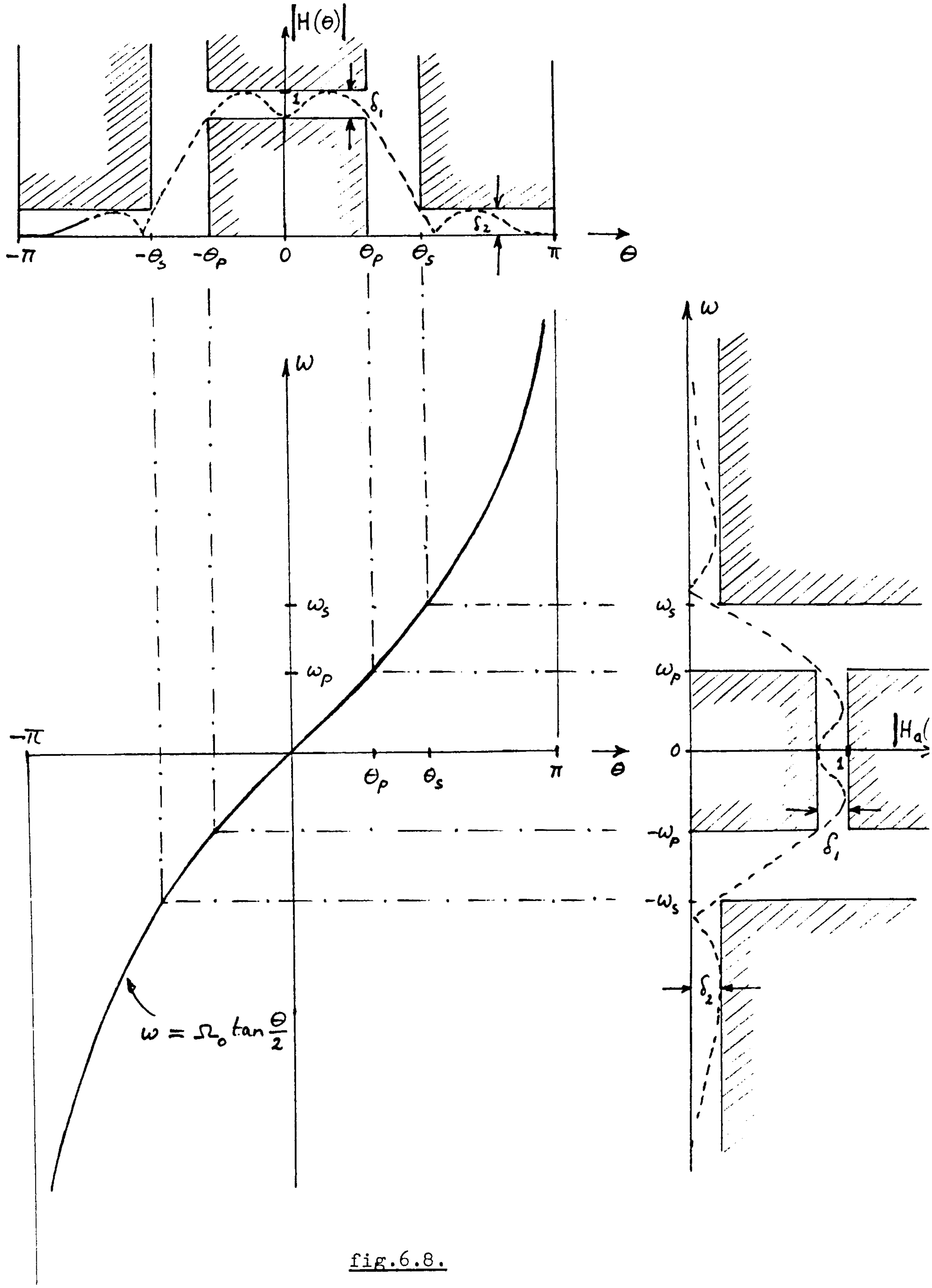

|H(θ)|
1
δ₁
δ₂
−π
-θs
-θp
0
θp
θs
π
θ
ω
ωs
ωp
−π
0
θp
θs
π
θ
-ωp
-ωs
δ₁
δ₂
|Ha(
1
ω = Ω₀ tan θ/2

fig.6.8.

When looking for such a filter the as yet not specified parameter $\Omega_o$ can be selected conveniently. In most books with filter tables the frequency is normalized such that $\omega_p = 1$ rad/sec.

Taking

$$\Omega_o = 1/ \tan \frac{\theta_p}{2} \tag{6.62}$$

it follows from (6.61) that $\omega_p$=1 rad/sec, which means that no normalization of the analog frequency axis is necessary to apply the tables.

Step 3. Find the location of the poles $\{p_i\}$ and zeros $\{q_i\}$ of the analog filter, and apply the bilinear transform. This gives the pole and zero locations of the digital filter:

$$\text{poles at} \quad \frac{\Omega_o+p_i}{\Omega_o-p_i} = \alpha_i$$

$$\text{zeros at} \quad \frac{\Omega_o+q_i}{\Omega_o-q_i} = \beta_i$$

Therefore the system function of the digital filter is:

$$\tilde{H}(z) = K.\frac{\prod_i (z-\beta_i)}{\prod_i (z-\alpha_i)} \tag{6.63}$$

and K can be found by setting $\tilde{H}(z)\Big|_{z=1} = \tilde{H}_a(p)\Big|_{p=o}$

In case the poles and zeros are not specified, one can take the system function $\tilde{H}_a(p)$ in any other suitable form and apply the bilinear transformation to obtain $\tilde{H}(z)$:

$$\tilde{H}(z) = \tilde{H}_a\left(\Omega_o \frac{z-1}{z+1}\right)$$

In the example we used a low-pass filter. It is known that other types of analog filters such as high-pass and band-pass filters can be obtained from a low-pass design by using a suitable transformation. These have been described in the course notes on filter design of Mr.Carriere. Such transformations can be used to advantage during step 2, when determining an analog filter satisfying the constraints in the $\omega$-domain.

In conclusion the bilinear transform is a viable technique for designing digital filters using all the existing powerful methods of analog filter design.
The resulting transmission function $H(\theta)$ is a warped version of the analog frequency response $H_a(\omega)$ due to the nonlinear relation between $\theta$ and $\omega$.
For small values of $\theta$ this relation is approximately linear: $\theta \approx (2/\Omega_o)\omega$, but especially for $\theta$ near $\pi$ the relation becomes very nonlinear. If we take $\Omega_o = 2/T$ then for small values of $\theta$ we have that $\theta \approx \omega T$ and this means that the frequency behavior of the analog and digital filters are quite close for low frequencies. Therefore this may be a convenient choice for $\Omega_o$ in certain cases, But when the analog filter is merely used as a reference filter in the filter design, then the value given in (6.62) is much more suitable.

## 6.3.5. Wave digital filters

In the methods discussed in the previous sections a certain system characterization like the impulse response, the system function or differential equation(s) of the analog system was transformed to a similar characterization in the digital domain. This means that in general with these methods there is no direct mapping of the structure and the corresponding parameters of the analog prototype filter and the resulting digital structure.

The basic idea behind the wave digital filter concept, which has been developed by Fettweis, is to make a mapping per element of an analog prototype filter into a digital structure. The advantage then is that in this way certain desirable properties of the analog filter can be retained in the digital filter. In particular, it is known that analog LC filters have a low sensitivity for variations of the element values (L and C values). In an element-wise transformation a low-sensitivity digital filter may then be obtained, which will have the advantage that its coefficients can be represented by a small number of bits.

### Design procedure

The design procedure starts with writing the "voltage-wave"-equations for an analog one-port (impedance Z). This one-port is depicted in fig. 6.9. (We use a tilde (~) for denoting analog signals in this

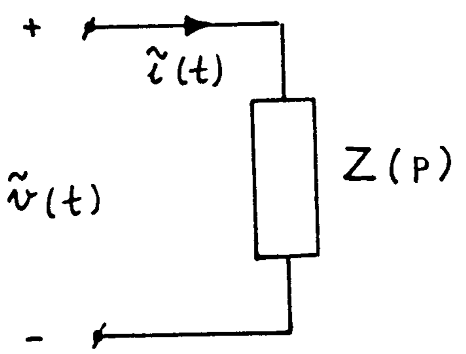



Fig. 6.9.

section). We define the wave variables:

$$\tilde{a}(t) = \tilde{v}(t) + R\,\tilde{i}(t) \qquad (6.64)$$

$$\tilde{b}(t) = \tilde{v}(t) - R\,\tilde{i}(t) \qquad (6.65)$$

where $\tilde{v}(t)$ is the voltage across, and $\tilde{i}(t)$ the current through the one-port. R is, for the moment being, an arbitrary normalization constant. $\tilde{a}(t)$ and $\tilde{b}(t)$ are called the incident (input) and reflected (output) waves, respectively.

Let us now consider the case that the one port is an inductor, i.e.

$$\tilde{v}(t) = L \frac{d\,\tilde{\imath}(t)}{dt} . \qquad (6.66)$$

Taking Laplace transforms yields

$$\tilde{V}(p) = pL\ \tilde{I}(p). \qquad (6.67)$$

Using (6.64), (6.65) and (6.67) we find for the Laplace transforms of the wave variables:

$$\tilde{B}(p) = \frac{pL-R}{pL+R}\ \tilde{A}(p) \qquad (6.68)$$

Defining $\Omega_o$ by

$$\Omega_o = R/L \qquad (6.69)$$

this gives:

$$\tilde{B}(p) = -\frac{\Omega_o - p}{\Omega_o + p}\ \tilde{A}(p) \qquad (6.70)$$

If we now apply the bilinear transform

$$Z \longleftrightarrow \frac{\Omega_o + p}{\Omega_o - p}$$

to these equations we find a relation between the corresponding digital variables A(Z) and B(Z) of the form

$$B(Z) = -Z^{-1}\ A(Z) \qquad (6.71)$$

which corresponds with a digital structure described by

$$b(n) = -a(n-1) .$$

Thus, with a wave-digital transformation, an inductor is to be replaced by a cascade of an inverter and a delay element, as shown in fig. 6.10a.

Similarly, if we repeat this procedure for a capacitor we find

$$\tilde{\imath}(t) = C \frac{d\,\tilde{v}(t)}{dt}$$

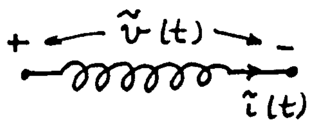
and hence

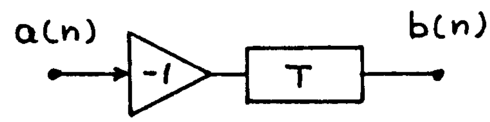
$$\tilde{B}(p) = \frac{\frac{1}{pC} - R}{\frac{1}{pC} + R}\ \tilde{A}(p) . \qquad (6.72)$$

## Wave digital transformations

**Analog prototype** | **Digital equivalent**

a) inductor



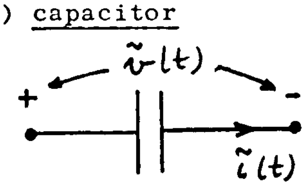

$$\tilde{v}(t) = L \frac{d\,\tilde{i}(t)}{dt}$$

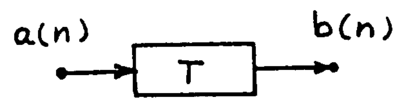
$$b(n) = -a(n-1)$$

$$\tilde{B}(p) = -\frac{R/L - p}{R/L + p}\,\tilde{A}(p)$$

$$B(Z) = -Z^{-1}\,A(Z)$$

b) capacitor



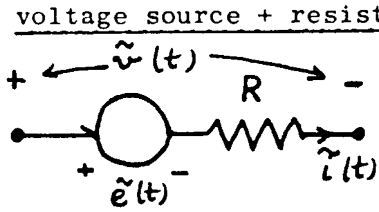

$$\tilde{i}(t) = C \frac{d\,\tilde{v}(t)}{dt}$$

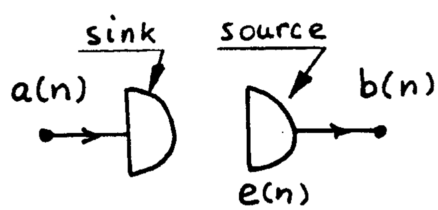
$$b(n) = a(n-1)$$

$$\tilde{B}(p) = \frac{\frac{1}{RC} - p}{\frac{1}{RC} + p}\,\tilde{A}(p)$$

$$B(Z) = Z^{-1}\,A(Z)$$

c) voltage source + resistor




$$\tilde{v}(t) = \tilde{e}(t) + R\,\tilde{i}(t)$$

$$b(n) = e(n)$$

$$\tilde{B}(p) = \tilde{E}(p)$$

$$B(Z) = E(Z)$$

If $\tilde{e}(t) = 0$ then only a resistor: $B(Z) = 0$

Fig. 6.10.

Now using

$$\Omega_o = \frac{1}{RC} \tag{6.73}$$

we find

$$\tilde{B}(p) = \frac{\Omega_o - p}{\Omega_o + p} \tilde{A}(p) \tag{6.74}$$

to which corresponds after bilinear transformation.

$$B(Z) = Z^{-1} A(Z) \tag{6.75}$$

and

$$b(n) = a(n-1) \tag{6.76}$$

Therefore a capacitor has to be replaced by a delay element (see fig. 10b).

Note that to obtain this result we had to take a different value for $\Omega_o$ than in the case of an inductor. We will come back to this point shortly. As a further important element we consider a voltage source with a series resistance R, shown in fig. 10c. For this element we have:

$$\tilde{v}(t) = R\,\tilde{\imath}(t) + \tilde{e}(t) \tag{6.77}$$

Hence we find

$$\tilde{B}(p) = \tilde{E}(p) \tag{6.78}$$

irrespective of the value of $\tilde{A}(p)$. The corresponding digital equivalent is characterized by

$$B(Z) = E(Z) \tag{6.79}$$

or

$$b(n) = e(n). \tag{6.80}$$

This is indicated by the symbols of a source and a sink, where the latter is an element that "absorbs" any incident signal. (We could also let the node open, but follow here the symbolism introduced by Fettweis). The structure is given in fig. 10c.

Eq. (6.78) was derived under the assumption that the normalization constant R for the voltage-wave formulation was equal to the value of the series resistance of the voltage source. In fact we have now fooled the situation three times. We are allowed to use only a single value for $\Omega_o$ in the bilinear transform, and we had to take the value of R equal to $\Omega_o L$ for the mapping of the inductor and equal to $1/\Omega_o C$ for that of the capacitor, while it should be equal to the value of the series resistance of the voltage source in the last example. In an arbitrary analog filter we will encounter all different values for R, L and C, so how do we proceed in that case ?

We then make use of so-called adaptors, which allow us to use different values of R for the wave equations of each element. To show how this works we consider the simple example of two elements in parallel, depicted in fig. 6.11a. We can use a normalization constant $R_1$ for the voltage waves of the inductor and a value $R_2$ for that of the capacitor, but then the interconnection between the two elements must be described by a somewhat more complicated set of wave equations. By inspection of fig. 6.11b we see that we have for this interconnection

$$\tilde{v}_1(t) = \tilde{v}_2(t) \qquad (6.81)$$

$$\tilde{i}_1(t) = -\tilde{i}_2(t). \qquad (6.82)$$

The corresponding wave equations are now given by

$$\tilde{a}_1(t) = \tilde{v}_1(t) + R_1\, \tilde{i}_1(t) \qquad (6.83)$$

$$\tilde{b}_1(t) = \tilde{v}_1(t) - R_1\, \tilde{i}_1(t) \qquad (6.84)$$

for port number 1, and

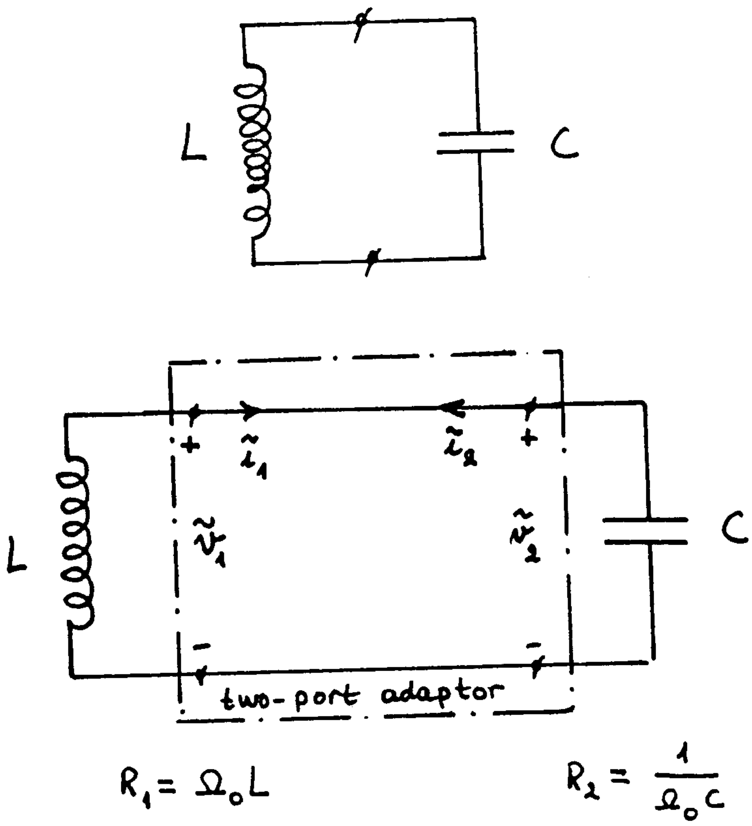



Fig. 6.11.

$$\tilde{a}_2(t) = \tilde{v}_2(t) + R_2\,\tilde{i}_2(t) \tag{6.85}$$

$$\tilde{b}_2(t) = \tilde{v}_2(t) - R_2\,\tilde{i}_2(t) \tag{6.86}$$

for port number 2. From (6.81) - (6.86) we then find after some manipulations:

$$\tilde{b}_1(t) = \tilde{a}_2(t) + \alpha(\tilde{a}_2(t) - \tilde{a}_1(t)) \tag{6.87}$$

$$\tilde{b}_2(t) = \tilde{a}_1(t) + \alpha(\tilde{a}_2(t) - \tilde{a}_1(t)) \tag{6.88}$$

where

$$\alpha = \frac{R_1 - R_2}{R_1 + R_2} \,. \tag{6.89}$$

Therefore to this two-port interconnection there corresponds a digital structure given by

$$b_1(n) = a_2(n) + \alpha(a_2(n) - a_1(n)) \tag{6.90}$$

$$b_2(n) = a_1(n) + \alpha(a_2(n) - a_1(n)) \tag{6.91}$$

which is depicted in fig. 6.12.

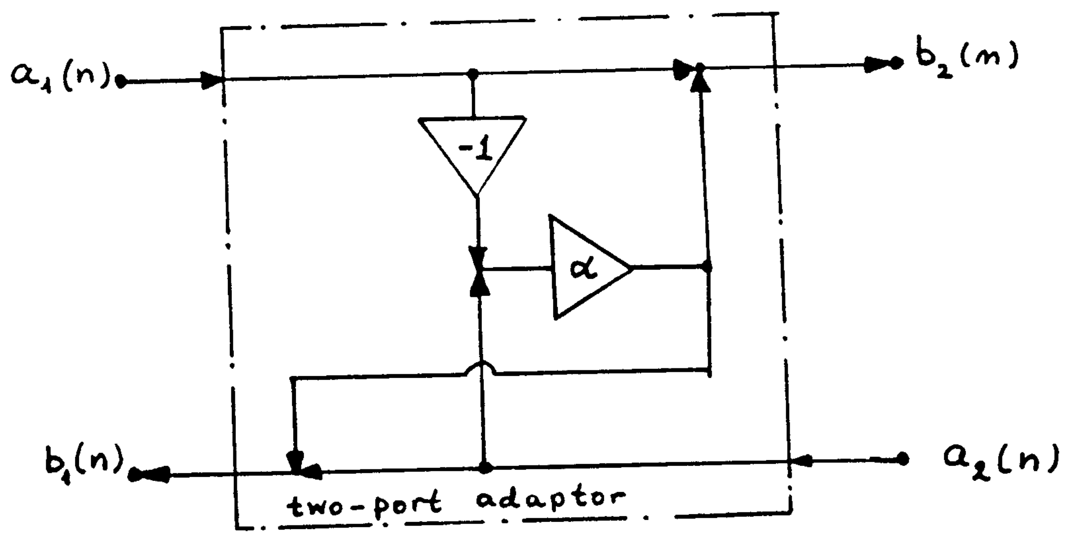



fig. 6.12.

We can proceed in a similar manner for interconnections of more than two elements. There are two different types of such interconnections, namely the series adaptor and the parallel adaptor, see fig. 6.13.

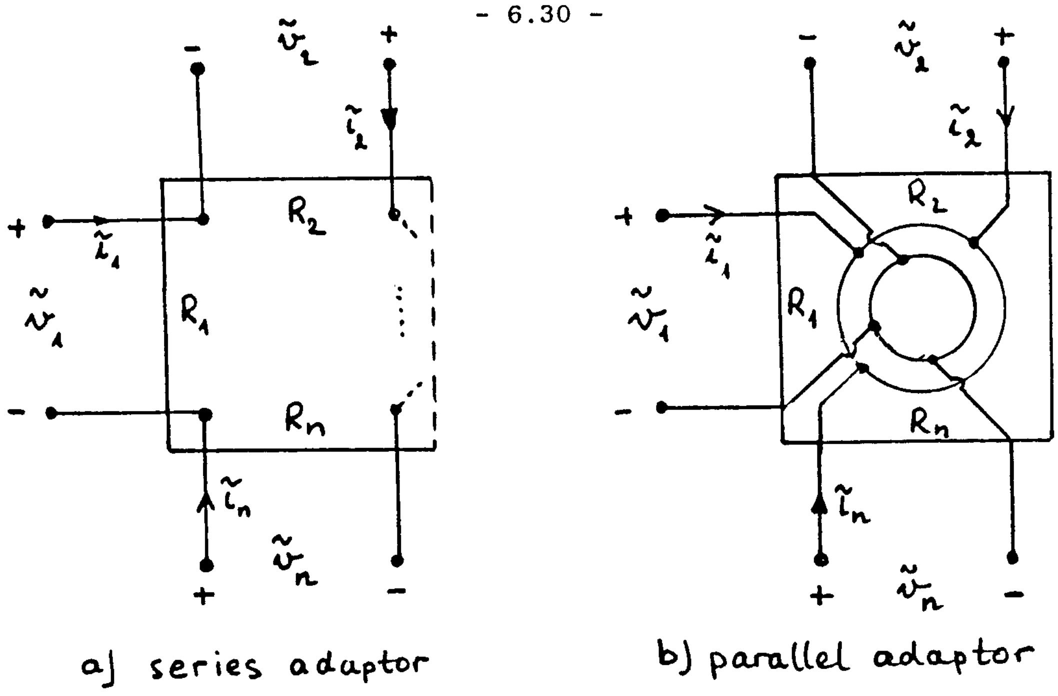



Fig. 6.13.

For the series adaptor we have

$$\tilde{v}_1(t) + \tilde{v}_2(t) + \ldots + \tilde{v}_n(t) = 0 \qquad (6.92)$$

$$\tilde{\imath}_1(t) = \tilde{\imath}_2(t) = \ldots = \tilde{\imath}_n(t) \qquad (6.93)$$

and for the parallel adaptor

$$\tilde{v}_1(t) = \tilde{v}_2(t) = \ldots = \tilde{v}_n(t) \qquad (6.94)$$

$$\tilde{\imath}_1(t) + \tilde{\imath}_2(t) + \ldots + \tilde{\imath}_n(t) = 0. \qquad (6.95)$$

Each of these equations leads to a corresponding set of equations between the wave variables and thus to a corresponding digital adaptor structure.

For example, a three-port parallel adaptor is given by

$$b_1 = (\alpha_1-1)\ a_1 + \alpha_2 a_2 + \alpha_3 a_3 \qquad (6.96)$$

$$b_2 = \alpha_1 a_1 + (\alpha_2-1)\ a_2 + \alpha_3 a_3 \qquad (6.97)$$

$$b_3 = \alpha_1 a_1 + \alpha_2 a_2 + (\alpha_3-1)\ a_3 \qquad (6.98)$$

and a three port series adaptor by

$$b_1 = (1-\beta_1)\ a_1 - \beta_1 a_2 - \beta_1 a_3 \qquad (6.99)$$

$$b_2 = -\beta_2 a_1 + (1-\beta_2) a_2 - \beta_2 a_3 \qquad (6.100)$$

$$b_3 = -\beta_3 a_1 - \beta_3 a_2 + (1-\beta_3) a_3 \qquad (6.101)$$

where $\alpha_k = 2G_k/(G_1+G_2+G_3)$, $G_k = 1/R_k$ and $\beta_k = 2R_k/(R_1+R_2+R_3)$.

The procedure to obtain a complete wave digital filter from a given analog prototype filter is now the following.

1) Split the filter into elements connected by series and parallel adaptors.

2) Transform each of the elements to its digital equivalent using the equivalences of fig. 6.10.

3) Transform each of the adaptors into digital adaptors given by eq. (6.96) - (6.101) (or corresponding equations for adaptors with more than three elements).

This procedure is indicated schematically in fig. 6.14 for a simple filter, but can be applied to any type of RLC analog prototype filter. It can be remarked that the delay elements correspond with the inductors and capacitors, while the multiplications stem from the transformation of the adaptors.

The transmission function of the analog prototype filter was given by the ratio of the Laplace transform of the voltage across the output resistance $R_5$ and the input voltage source $\tilde{e}_1$. It is clear from the foregoing analysis that the transmission function of the wave digital filter, which is the ratio of the Z-transform of $b_5$ and $e_1$ will be related to that of the analog filter by means of the bilinear transform. Hence, given the specifications of the digital filter one first has to transform these via this transform to specifications for the analog filter just as was the case in section 6.3.4.

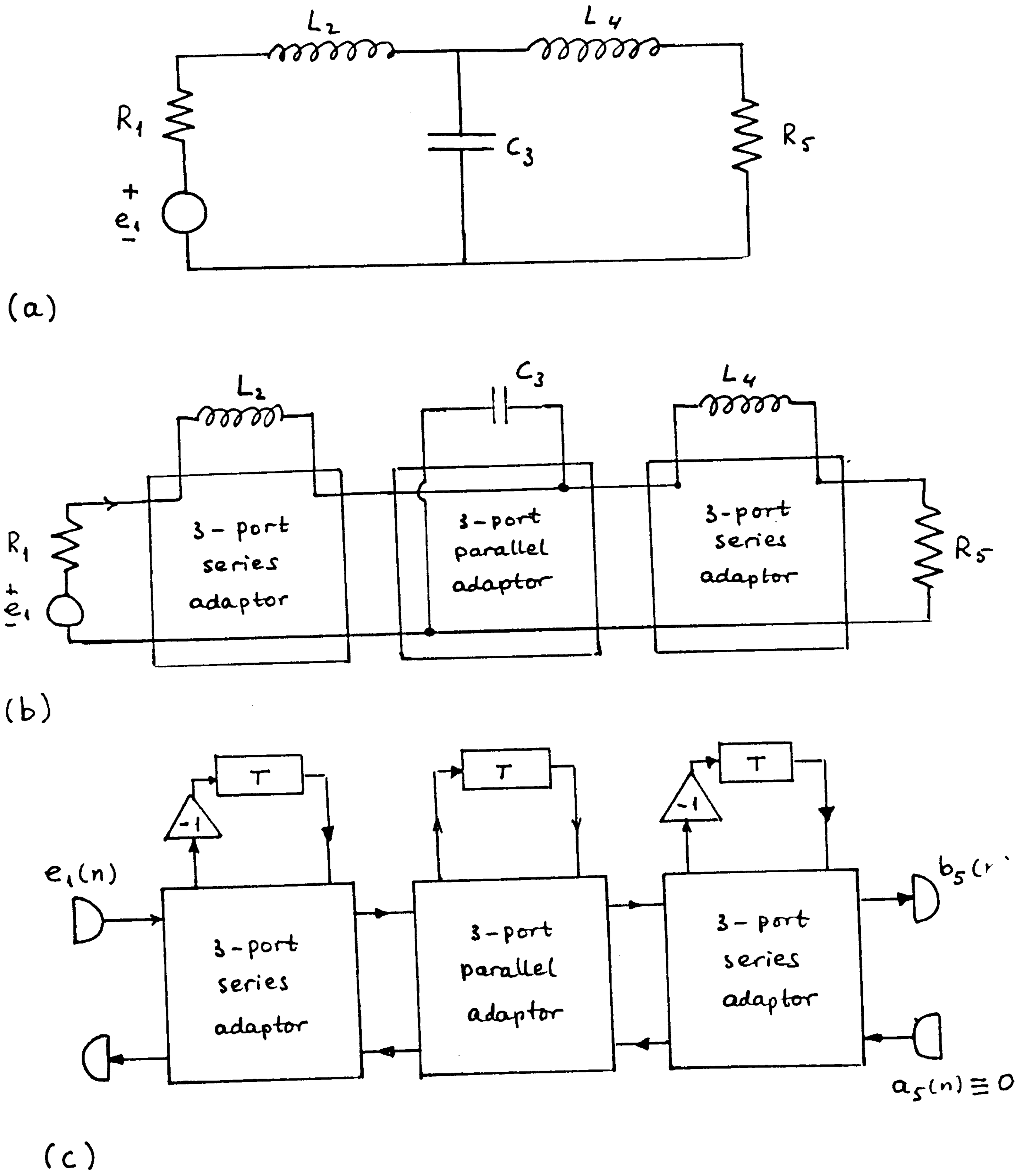

L2
L4
R1
C3
R5
+
e1
−
(a)
L2
C3
L4
R1
+
e1
−
3-port
series
adaptor
3-port
parallel
adaptor
3-port
series
adaptor
R5
(b)
T
T
T
-1
-1
e1(n)
3-port
series
adaptor
3-port
parallel
adaptor
3-port
series
adaptor
a5(n) ≡ 0
(c)

Fig. 6.14.