3. Linear shift-invariant discrete systems

A discrete system H is an (implementation of an) algorithm that maps a discrete input signal $x(n)$ into a discrete output sigal $y(n)$ called its response. The actual realization of this system may have many different forms; it may be a set of TTL gates, a micro-processor with suitably programmed ROM, or a software package in a general purpose computer. For the description that follows the nature of the implementation is not of importance, but the nature of the algorithm is. We will confine ourselves for the moment to linear shift-invariant systems to be defined now.

Let $x(n)$, $x_1(n)$ and $x_2(n)$ be discrete time signals that are applied to a discrete system H, and let this system respond to these signals by producing the output signals $y(n)$, $y_1(n)$ and $y_2(n)$, respectively. This will be denoted by:

$$x(n) \xrightarrow{H} y(n)$$

$$x_1(n) \xrightarrow{H} y_1(n)$$

$$x_2(n) \xrightarrow{H} y_2(n)$$

The system H is <u>linear</u> if and only if for every x_1 and x_2 and every α_1 and α_2 we have that

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \xrightarrow{H} \alpha_1 y_1(n) + \alpha_2 y_2(n) \tag{3.1}$$

From the special case $x_1(n) = x_2(n) \equiv 0$ it follows that for all linear systems

$$x(n) \equiv 0 \xrightarrow{H} y(n) \equiv 0 \tag{3.2}$$

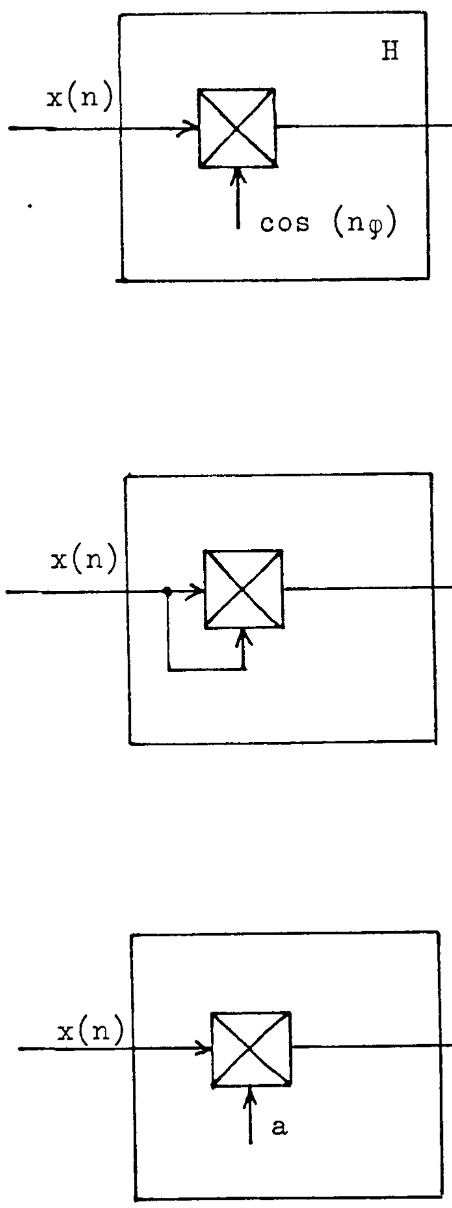
which means that a linear system cannot generate an output signal without being excited.

The system H is <u>shift</u> (or time) - invariant if for every $x(n)$ and n_o

$$x(n-n_o) \xrightarrow{H} y(n-n_o) \tag{3.3}$$

which means that the system is invariant under a shift of the time axis.

As an example three simple digital systems are shown in fig. 3.1 consisting only of a multiplier, and it can be seen that depending on the particular input signals of the multiplier the system can or cannot be linear, and shift invariant.



$y(n) = x(n) \cdot \cos n\varphi$ $(\varphi \neq k \cdot 2\pi)$

linear, but not shift invariant

$y(n) = x^2(n)$

non linear, shift invariant

$y(n) = a \cdot x(n)$

linear, shift invariant

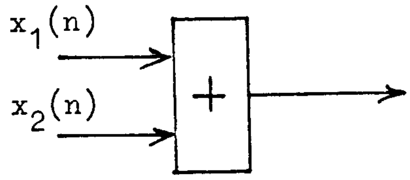
Fig. 3.1.

In this section we will deal only with linear shift-invariant systems.

3.1. Elements

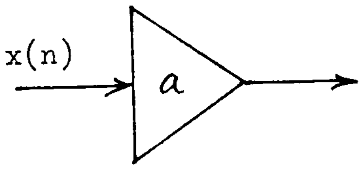
The systems that will be encountered can all be constructed from the following three elements.

a) adder.



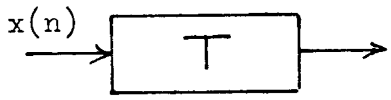
$y(n) = x_1(n) + x_2(n)$

b) (constant) multiplier.



$y(n) = a \cdot x(n)$

c) unit delay.



$y(n) = x(n-1)$

3.2. Difference equations

A system consisting of these elements can always be described by a set of difference equations. As an example consider the system in fig. 3.2. By inspection we can write:

$$v(n) = a.x(n) + b.x(n-1) + y(n)$$
$$y(n) = c \cdot v(n-1) \tag{3.4}$$

Strictly speaking a difference equation should contain entities of the form $\Delta x(n), \Delta y(n)$, etc. where $\Delta x(n) = x(n) - x(n-1)$, $\Delta y(n) = y(n) - y(n-1)$. It is not difficult to bring eq. (3.4) in such a form (do it!) but for our purpose it is not necessary, and it is common practice to call equations of the form (3.4) difference equations as well.

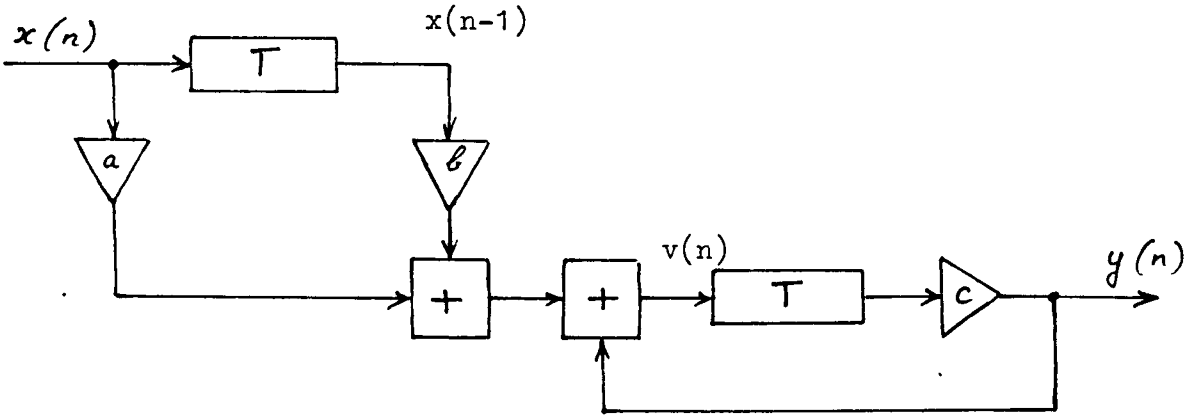


Fig. 3.2.

Eqns (3.4) enable us to determine the output signal y(n) for a given input signal x(n). Let us, for example, take $x(n) = \delta(n)$, and assume $v(n) = 0$ for $n < 0$. Then we find:

$$y(n) = 0 \qquad n \leq 0$$

$$v(0) = a \longrightarrow y(1) = a \cdot c$$

$$v(1) = b + a \cdot c \longrightarrow y(2) = (b + a \cdot c)\, c$$

$$v(2) = bc + ac^2 \longrightarrow y(3) = (bc + ac^2)c$$

$$\vdots$$

$$v(n-1) = bc^{n-2} + ac^{n-1} \longrightarrow y(n) = bc^{n-1} + ac^n \qquad n \geq 2$$

This response to the impulse excitation has been given the name impulse response and it is a very important characteristic of a discrete system.

3.3. The impulse response

To see how important the impulse response is, consider again the system in fig. 3.2. With some trouble we have found the response for the case $x(n) = \delta(n)$. But what to do if it is excited by a different input signal. Do we have to start all over again ? Let us denote the impulse response by h(n). Then clearly from the fact that we know the system to be linear we know that excitation with $x(0) \cdot \delta(n)$ must result in the response $y(n) = x(0) \cdot h(n)$.

Moreover, since the system is shift invariant the response to $\delta(n-k)$ must be $y(n) = h(n-k)$, whatever value k may have.

Now recall eq. (1.7) which states that each x(n) can be written as a weighted sum of shifted unit impulses. Then because of the linearity and shift invariance it follows that

$$\delta(n) \xrightarrow{H} h(n)$$

$$\delta(n-k) \xrightarrow{H} h(n-k)$$

$$x(k)\ \delta(n-k) \xrightarrow{H} x(k)\ h(n-k)$$

$$\cdot \sum_{k=-\infty}^{\infty} x(k)\ \delta(n-k) \xrightarrow{H} \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$

$$x(n) \xrightarrow{H} y(n) = x(n) * h(n) \qquad (3.5)$$

which means that the response to an arbitrary input signal x(n) is the convolution of this signal with the impulse response of the system.

To obtain some experience with this type of computation determine for the system in fig. 3.3 the impulse response, and the response

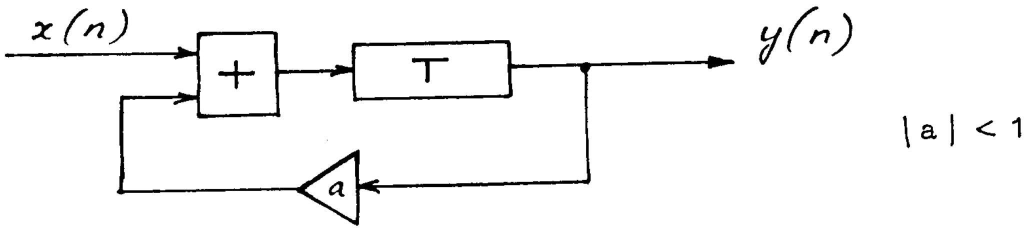


Fig. 3.3.

to x(n) = A cos n θ.

Apart from the fact that it enables us to compute the response of the system to arbitrary input signals, the impulse response provides us with some additional information that will be discussed now.

3.4. Causality

An important aspect of all physically realizable systems is causality.

Definition. A system is causal if and only if the response at any particular time n is not dependent on input signal values occurring at times later than n.

With this definition it is not difficult to imagine why causality is so important. It will take some time to figure out a realistic system that can respond on input signal values that still have to be applied to it. (They do exist, however; think of a magnetic recording system where the magnetic head is already "sensing" the magnetic field produced by parts of the tape that are close to it but not yet have reached the head.)

Proposition 1. A linear shift-invariant discrete system is causal if and only if its impulse response is zero for $n < 0$.

Proof. Causality was defined as the property that $y(n)$ is only determined by $x(k)$ with $k \leq n$.
Now let $h(n) \neq 0$ for some $n_o < 0$. Then with the specific input signal $x(n) = x(0)\,\delta(n)$ we find $y(n) = x(0)\,.\,h(n)$ and in particular $y(n_o) = x(0)\,.\,h(n_o) \neq 0$ which means that at $n_o < 0$ the output is determined by $x(0)$, which is a value that at time n_o has not yet been applied to the system and thus the system is not causal.

On the other hand let $h(n) = 0 \quad n < 0$, then for any input signal $x(n)$ the response is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\; x(n-k)$$

$$= \sum_{k=0}^{\infty} h(k)\; x(n-k)$$

and clearly $y(n)$ is only determined by the present and past values of x.

The notion of causality is closely related to that of realizability. A non-causal system cannot be realized for real-time signal processing. It should be kept in mind, however, that in case of off-line processing causality is not really a problem. In that case even a time reversal can be accomplished which surely turns a causal system into a non-causal one.

3.5. Stability

A second important point of which the impulse response can provide the information is the stability.

Definition. A linear shift invariant discrete system is stable if and only if for every bounded input the response is bounded too:

$$\forall x(n);\ |x(n)| < M_x \xrightarrow{H} y(n)\ ;\ |y(n)| < M_y$$

Proposition 2. A linear shift invariant discrete system is stable if and only if there exists a number M_h such that

$$\sum_{n=-\infty}^{\infty} |h(n)| < M_h \qquad (3.6)$$

Proof. First assume that $\sum_n |h(n)|$ does not converge. Then take

$$x(n) = \text{sign}\ (h(-n)) = \begin{cases} 1 & h(-n) \geq 0 \\ -1 & h(-n) < 0 \end{cases}$$

This gives:

$$y(0) = \sum_{k=-\infty}^{\infty} x(-k)\ h(k) = \sum_{k=-\infty}^{\infty} |h(k)|$$

which by assumption is not bounded, thus proving the "only if" part.

Next assume (3.6) to be satisfied. Then from

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\ h(k)$$

it follows that:

$$|y(n)| \leqslant \sum_{k=-\infty}^{\infty} |x(n-h)|\ |h(k)|$$

and for every $x(n)$ with $|x(n)| < M_x$

$$|y(n)| \leqslant M_x \sum_{k=-\infty}^{\infty} |h(k)| < M_x\ M_h$$

which completes the proof.

Stability is not only important to assure that the output will be bounded, but also guarantees that the output will vanish if the input is set to zero from some time on, which means that transients will always have a decaying character.

This can more formally be expressed in

<u>Proposition 3</u>. If in a stable linear system $|x(n)| < M_x$ and for $n > n_o$ $x(n) \equiv 0$ then $\lim\limits_{n \to \infty} y(n) = 0$.

<u>Proof</u>. Consider

$$y(n_o+N) = \sum_{k=-\infty}^{\infty} h(k)\ x(n_o+N-k)$$

$$= \sum_{k=N}^{\infty} h(k)\ x(n_o+N-k)$$

Then

$$|y(n_o+N)| \leq M_x \sum_{k=N}^{\infty} |h(k)|$$

and since H is stable we have that $\sum\limits_{k=-\infty}^{\infty} |h(k)|$ converges to a finite value from which it follows that

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} |h(k)| = 0$$

Therefore $\lim_{N\to\infty} |y(n_0+N)| = 0$ which means that

$$\lim_{n\to\infty} y(n) = 0 .$$

Some examples of causal and non-causal, stable and non-stable impulse responses are given in table 3.1.

causal	stable	h(n)
yes	yes	$2^{-n}\ u(n)$
yes	no	$2^{n}\ u(n)$
no	yes	$2^{n}\ u(-n) + 2^{-n}\ u(n)$
no	no	$2^{-n}\ u(-n)$

Table 3.1.

3.6. Transmission function

Since the impulse response h(n) is a discrete function we can take its FTD to obtain the function $H(\theta)$:

$$H(\theta) = \sum_{n=-\infty}^{\infty} h(n)\ e^{-jn\theta} . \tag{3.7}$$

This function will be called the transmission function of the system, since it provides an immediate measure of how a sinusoidal signal is transmitted through the system. To see this let us compute the response of the system to the signal

$$x(n) = A \cdot \cos\ (n\theta + \varphi) .$$

This response is given by:

$$y(n) = x(n) * h(n)$$

$$= A \sum_{k=-\infty}^{\infty} \cos\ ((n-k)\ \theta + \varphi)\ h(k)$$

$$= \frac{A}{2}\ e^{j(n\theta + \varphi)} \sum_{k=-\infty}^{\infty} e^{-jk\theta}\ h(k)$$

$$+ \frac{A}{2}\ e^{-j(n\theta + \varphi)} \sum_{k=-\infty}^{\infty} e^{jk\theta}\ h(k)$$

and a comparison with eq. (3.7) gives:

$$y(n) = \frac{A}{2} \cdot e^{j(n\theta + \varphi)}\, H(\theta) + \frac{A}{2}\, e^{-j(n\theta + \varphi)}\, H(-\theta)$$

Since h(n) is assumed to be real valued it follows from table 1.1 that $H(-\theta) = H^{*}(\theta) = |H(\theta)|\, e^{-j \arg H(\theta)}$. Therefore

$$y(n) = A.\, |H(\theta)| \cos\,(n\theta + \varphi + \arg H(\theta)) \qquad (3.8)$$

and we see that $|H(\theta)|$ describes the attenuation at the relative frequency θ and arg(H(θ)) the phase shift between the input and the output signal in case of sinusoidal excitation.

For a more general type of excitation the transmission function determines the output spectrum by the relation:

$$Y(\theta) = H(\theta)\ .\ X(\theta) \qquad (3.9)$$

which follows immediately from (3.5) and the properties of the FTD.

From (3.9) the two equations

$$|Y(\theta)| = |H(\theta)|\ |X(\theta)|$$

$$\arg Y(\theta) = \arg X(\theta) + \arg H(\theta)$$

can be derived, and we see that $|H(\theta)|$ describes by how much the input spectrum is attenuated, whereas arg H(θ) gives information about the phase shift.

For a given system there are three ways to determine the transmission function:
1) from the impulse response by means of eq. (3.7)
2) from excitation with a cosine using eq. (3.8)
3) from the differential equations applying properties of the FTD and eq. (3.9).

This will be illustrated by means of the example of fig. 3.2.

ad 1

For the system in fig. 3.2 we have determined the impulse response to be:

$$h(n) = \begin{cases} 0 & n \leq 0 \\ a\ c & n = 1 \\ b\ c^{n-1} + a\ c^{n} & n \geq 2 \end{cases}$$

Therefore with eq. (3.7) we find:

$$H(\theta) = a.c\, e^{-j\theta} + \sum_{n=2}^{\infty} (b+a\,c)\, c^{n-1}\, e^{-jn\theta}$$

$$= a.c\, e^{-j\theta} + (b + a\,c)\, \frac{c\, e^{-j2\theta}}{1 - c\, e^{-j\theta}} \qquad |c| < 1$$

$$= c\, .\, \frac{a\, e^{-j\theta} + b\, e^{-j2\theta}}{1 - c\, e^{-j\theta}}$$

ad 2

Let $x(n) = A \cos n\,\theta$. Then $x(n-1) = A \cos (n-1)\,\theta$. We know that for this excitation $y(n) = B\, .\, A \cos (n\theta + \varphi)$ where $B = |H(\theta)|$, and $\varphi = \arg H(\theta)$.
Moreover, since; $y(n) = c\, v(n-1)$ we have
$v(n) = \frac{B}{c} A \cos((n+1)\,\theta + \varphi) = \frac{B}{c} A \cos (n\theta + \varphi + \theta)$

From eq. (3.4) it follows that:

$$\frac{B}{c}\, .\, A \cos (n\theta + \varphi + \theta) - B\, .\, A \cos (n\theta + \varphi) =$$
$$A\, (a \cos n\,\theta + b \cos (n-1)\,\theta)$$

or

$$B \left[\cos n\,\theta \cos (\varphi + \theta) - \sin n\,\theta \sin (\varphi + \theta)\right]$$
$$-B\, c \left[\cos n\,\theta \cos \varphi - \sin n\,\theta \sin \varphi\right]$$
$$= c\, .\, \left[a \cos n\,\theta + b \cos n\,\theta \cos \theta + b \sin n\,\theta \sin \theta\right]$$

Thus

$$\cos n\,\theta \left[B \cos \varphi \cos \theta - B \sin \varphi \sin \theta - B\, c \cos \varphi - a\, c - b\, c \cos \theta\right]$$
$$= \sin n\,\theta \left[B \sin \varphi \cos \theta + B \cos \varphi \sin \theta - B\, c \sin \varphi + b\, c \sin \theta\right]$$

This latter equation can only be satisfied for all n if

$$B \cos \varphi\, (\cos \theta - c) - B \sin \varphi\, .\, \sin \theta = a\, c + b\, c \cos \theta$$

$$B \cos \varphi \sin \theta + B \sin \varphi\, (\cos \theta - c) = -b\, c \sin \theta$$

These two equations have the solution:

$$B \cos \varphi = \frac{1}{1 + c^2 - 2\, c \cos \theta}\, .\, (\cos \theta - c)(a\, c + b\, c \cos \theta) = \mathrm{Re}\left[H(\theta\right.$$

$$B \sin \varphi = \frac{-1}{1 + c^2 - 2\, c \cos \theta} \sin \theta\, .\, b\, c \sin \theta \qquad = \mathrm{Im}\left[H(\theta\right.$$

From these equations $B = |H(\theta)|$ and $\varphi = \arg H(\theta)$ can be derived easily.

We see that this computation is rather cumbersome even for such a relatively simple network. This method for determining the transmission function is therefore in general not very appropriate. However, it should be kept in mind that this method does enable a technique for measuring $H(\theta)$ i.e. exciting the network, of which the precise structure then needs not to be known, by $x(n) = A \cos n\theta$ and measuring the output amplitude and phase.

ad 3

Applying the Fourier transform to eq. (3.4) gives:

$$V(\theta) = a\,X(\theta) + b\,X(\theta)\,e^{-j\theta} + Y(\theta)$$

$$Y(\theta) = c\,V(\theta)\,e^{-j\theta}$$

from which it follows:

$$Y(\theta)\,(1 - c\,e^{-j\theta}) = (a\,c\,e^{-j\theta} + b\,c\,e^{-j2\theta})\,X(\theta)$$

Thus:

$$Y(\theta) = X(\theta)\;.\;c\,\frac{a\,e^{-j\theta} + b\,e^{-j2\theta}}{1 - c\,e^{-j\theta}} = X(\theta)\;.\;H(\theta)$$

Clearly, this is a very simple and efficient way to determine $H(\theta)$ analytically and can be applied to rather complex systems without much difficulty.

3.7. The system function

The system function of a discrete system H is defined as the z-transform of the impulse response:

$$\tilde{H}(z) = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n} \tag{3.10}$$

This function is related to the transmission function by eq. (1.22):

$$\tilde{H}(e^{j\theta}) = H(\theta) \tag{3.11}$$

which means that the transmission function is equal to the system function evaluated on the unit circle of the z-plane.

Values of z where $H(z) = 0$ will be called zeroes of the system function, and values where $H(z) = \infty$ will be called poles.

As was already remarked before, we will only deal with rational functions of z, and these functions are, except for a gain factor α, fully determined by a specification of their poles $\left\{z = p_k\right\}_{k=1}^{N}$ and zeroes $\left\{z = z_k\right\}_{k=1}^{M}$

$$\tilde{H}(z) = \alpha \cdot \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)} \tag{3.12}$$

$$= \frac{\sum_{i=0}^{M} a_i \, z^i}{\sum_{j=0}^{N} b_j \, z^j}$$

The coefficients a_i, and b_j are directly related to the poles and zeroes of the system function.

In all cases these coefficients a_i and b_j will be real valued, which implies that poles and zeroes are either real or they occur in complex conjugate pairs.

It is possible by a partial fraction expansion of $\tilde{H}(z)$ to rewrite eq. (3.12) in the form:

$$\tilde{H}(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k \, z^{-1}} + A_o \tag{3.13}$$

(where it is assumed that N = M). After a comparison with eq. (1.16) we see that the corresponding impulse response is given by:

$$h(n) = \sum_{k=1}^{N} A_k \cdot p_k^{\,n} \cdot u(n) + A_o \, \delta(n). \tag{3.14}$$

From eq. (3.14) it is possible to determine whether or not the syster that has this impulse response will be stable. For this we must determine

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} \left\{ \left| \sum_{k=1}^{N} A_k \, p_k^{\,n} + A_o \delta(n) \right| \right\}$$

$$\leq \sum_{n=0}^{\infty} \left\{ \left| \sum_{k=1}^{N} A_k \, p_k^{\,n} \right| + \left| A_o \delta(n) \right| \right\}$$

$$\leq |A_o| + \sum_{n=0}^{\infty} \sum_{k=1}^{N} \left| A_k \, p_k^{\,n} \right|$$

$$= |A_o| + \sum_{n=0}^{\infty} \sum_{k=1}^{N} |A_k| \, |p_k|^n$$

$$= |A_o| + \sum_{k=1}^{N} \sum_{n=0}^{\infty} |A_k| |p_k|^n$$

$$= |A_o| + \sum_{k=1}^{N} |A_k| \sum_{n=0}^{\infty} |p_k|^n$$

$$= |A_o| + \sum_{k=1}^{N} \frac{|A_k|}{1-|p_k|}$$

if $|p_k| < 1$ for $k = 1, 2, \ldots, N$.

Thus the system will be stable if all poles have a magnitude less than one, which means that all poles lie inside the unit circle $|z| = 1$.

If, on the other hand one of the poles lies on or outside the unit circle, then for this pole

$$\lim_{n \to \infty} |p_k|^n \neq 0$$

which implies that $\sum_{n=0}^{\infty} |h(n)| \longrightarrow \infty$.

Therefore we can conclude that a <u>necessary and sufficient condition for stability</u> of a discrete system is that <u>the poles of the system function all lie inside the unit circle of the z-plane</u>.