2. Analog to digital and digital to analog conversion

2.1. Introduction

Thus far we have not bothered where the digital signals came from. However, the whole theory of digital signals is meant to deal with practical situations of signal processing. Although in an increasing number of applications the signals are generated by digital generators (structures for generators of for example sine-oscillators will be discussed later) it cannot be denied that at present most signals originate from an analog source, and at some time have to be converted again into analog signals too. The interfaces between analog and digital systems are formed by analog to digital (A/D) and digital to analog (D/A) converters. It goes far beyond the scope of this course to indicate explicitly how the process of A/D and D/A conversion takes place in all details. Rather we will concentrate on finding a description of these devices both in the time-domain and the frequency domain that will enable us to deal with systems consisting of both analog and digital devices.

An analog signal in the time domain is a continuous function of the time parameter t, and a frequency domain description may be obtained from the FTC.

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\omega t}\, dt \qquad (2.1)$$

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega)\, e^{j\omega t}\, d\omega \qquad (2.2)$$

A digital signal is a function of the discrete time parameter n and has a spectral description by means of the FTD defined by eqns (1.11), (1.12).

The practical A/D converter is a physical device that performs the conversion. The A/D and D/A converter we will consider here will be more or less abstract elements, that connect the two worlds via a set of relations between the corresponding time- and frequency-domain representations of the signals.They are linear devices, that operate ideally. It is possible with these idealized devices, however, to model practical circuits for A/D and D/A conversion by extending them with analog and/or digital circuitry, and examples of this will be given.

2.2. Analog to digital converter

For an A/D converter we will use throughout the symbol depicted in fig.2.1.

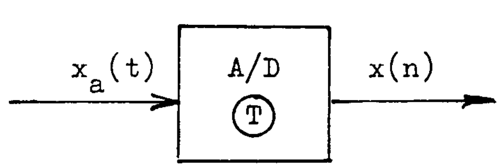

Fig.2.1.

It has an analog input signal $x_a(t)$ and a digital output signal $x(n)$ and the operation it performs is defined by[x]) :

$$x(n) = x_a(nT) \tag{2.3}$$

It is thus characterized by a single parameter T designated as <u>sampling period.</u>

The corresponding frequency domain relations can be derived as follows.
From eq.(2.2) we have

$$x_a(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\omega)\, e^{j\omega nT}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{k.2\pi-\pi}^{k.2\pi+\pi} X_a(\omega)\, e^{j\omega nT}\, \frac{d\omega T}{T}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{T} \int_{-\pi}^{\pi} X_a(\omega+k\,\tfrac{2\pi}{T})\, e^{j(\omega+k\,\frac{2\pi}{T})nT}\, d\omega T$$

$$x_a(nT) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega+k\,\tfrac{2\pi}{T}) \right\} e^{jn\omega T}\, d\omega T \quad (2$$

Since also

$$x_a(nT) = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta)\, e^{jn\theta}\, d\theta \tag{2.5}$$

it can be concluded from (2.4) and (2.5) that

$$X(\omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\,(\omega+k.\tfrac{2\pi}{T}) \tag{2.6}$$

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x) The subscript a will be used to indicate that the corresponding signal is analog. Using this convention $\delta_a(t)$ will denote the Dirac-function that must well be distinguished from the unit-impulse $\delta(n)$ defined by (1.1)

Eq.(2.5) is the desired frequency domain description, which relates the spectrum of the output signal of the A/D converter to that of the input signal.
Eq.(2.6) is expressed in terms of the frequency parameter ω used in the analog description (FTC). A similar expression in terms of the relative frequency parameter θ of the FTD is

$$X(\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a((\theta+k2\pi)/T) \qquad (2.7)$$

and we see that the relation between ω and θ is given by

$$\theta = \omega T \qquad (2.8)$$

(Eq.(2.8) explains the term relative frequency: it is the frequency relative to the sampling frequency in case the digital signal is obtained from an analog signal by sampling with period T).
Eq.(2.6) or (2.7) has the following interpretation:
the A/D conversion folds back all parts of the spectrum of the analog signal from $-\infty < \omega < \infty$ to the fundamental interval of the spectrum of the digital signal. This is illustrated by an example in Fig.2.2.

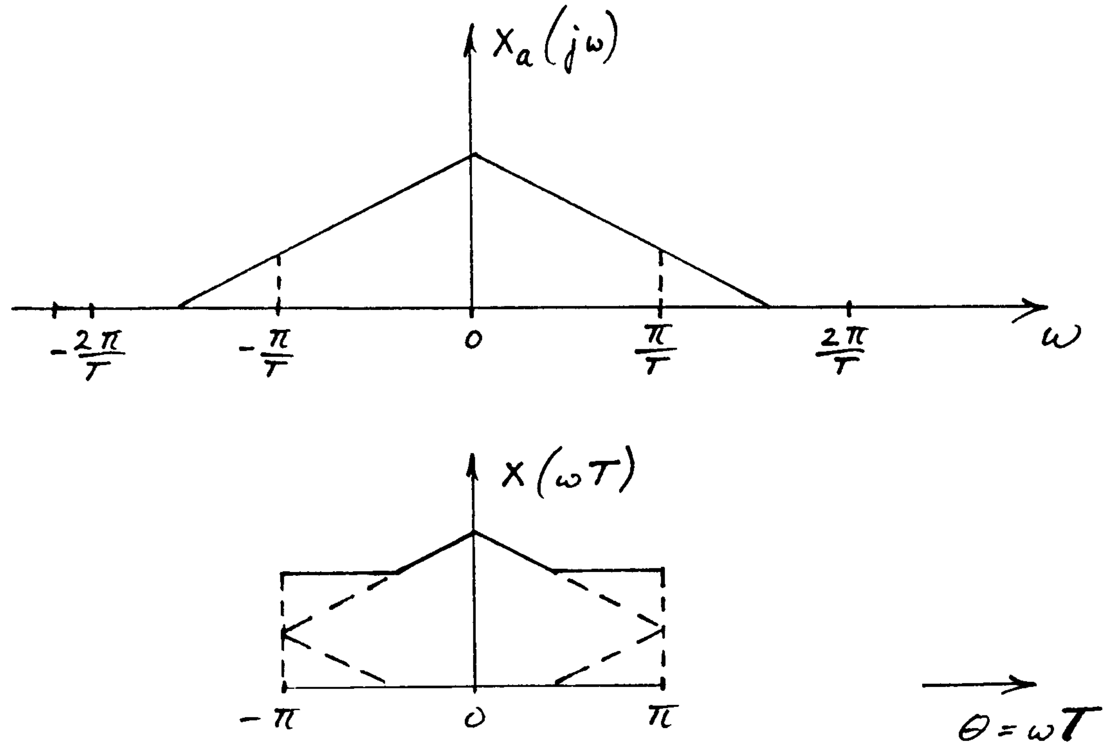

Fig.2.2.

In that example the sum in (2.6) will only contain three non zero terms for $|\omega T| \leq \pi$ $(k=0, \pm 1)$ since $X_a(\omega)$ is bandlimited to $3\pi/2T$. All terms that give contributions to the spectrum of X due to folding are called aliasing.

Eq.(2.6) tells us therefore that aliasing can only be avoided if the analog signal is bandlimited :

$$X_a(\omega) = 0 \qquad |\omega| \geq \frac{\pi}{T} \tag{2.9a}$$

In that case:

$$X(\omega T) = \frac{1}{T} X_a(\omega) \qquad \text{for } -\pi/T < \omega < \pi/T \tag{2.9b}$$

2.3. A practical A/D converter

A practical A/D converter often operates in the following way. First the analog signal $x_a(t)$ is applied to a sample and hold circuit that delivers the signal $y_a(t)$ that is approximately constant over at least some time τ after every sampling (see fig.2.3) $(\tau \leq T)$

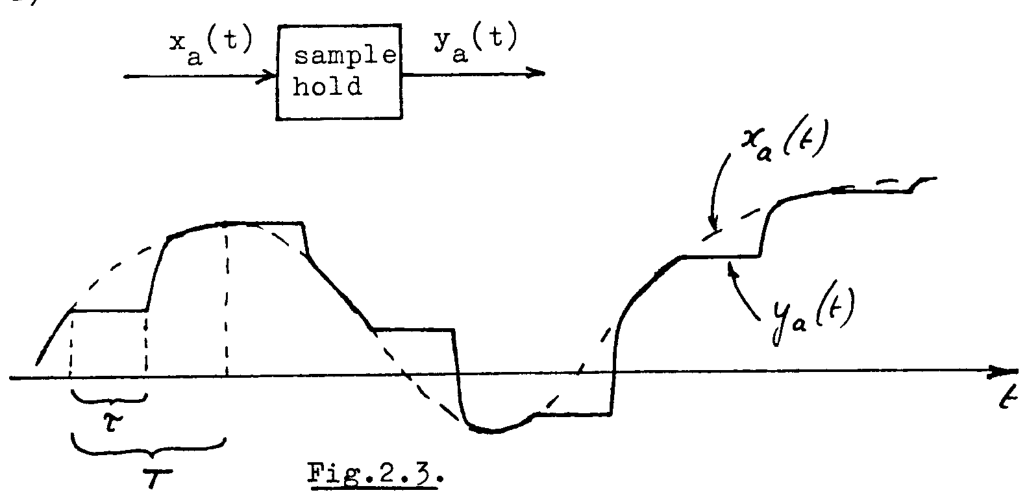

Fig.2.3.

$y_a(t)$ is then applied to a tree of comparators that during the time τ compare the value of y_a with a set of references that are related by powers of two.
When all comparisons are made the decision circuits are enabled by a clock and deliver the output signal as a number in binary format, the bits coming out in parallel.
The whole procedure can thus be modelled as indicated in fig.2.4

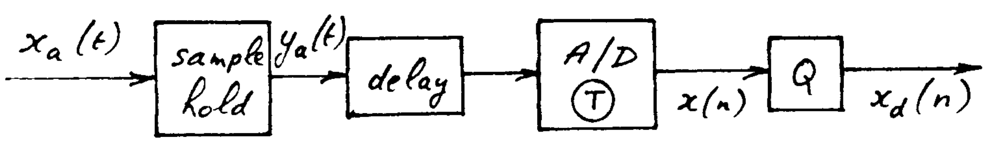

Fig.2.4.

where Q is a quantization nonlinearity with a characteristic as shown in fig.2.5. This nonlinearity accounts for the fact that the signal is represented only with a finite number of bits. If the number of bits is equal to b then $\hat{x} = 2^{b-1}.q$.

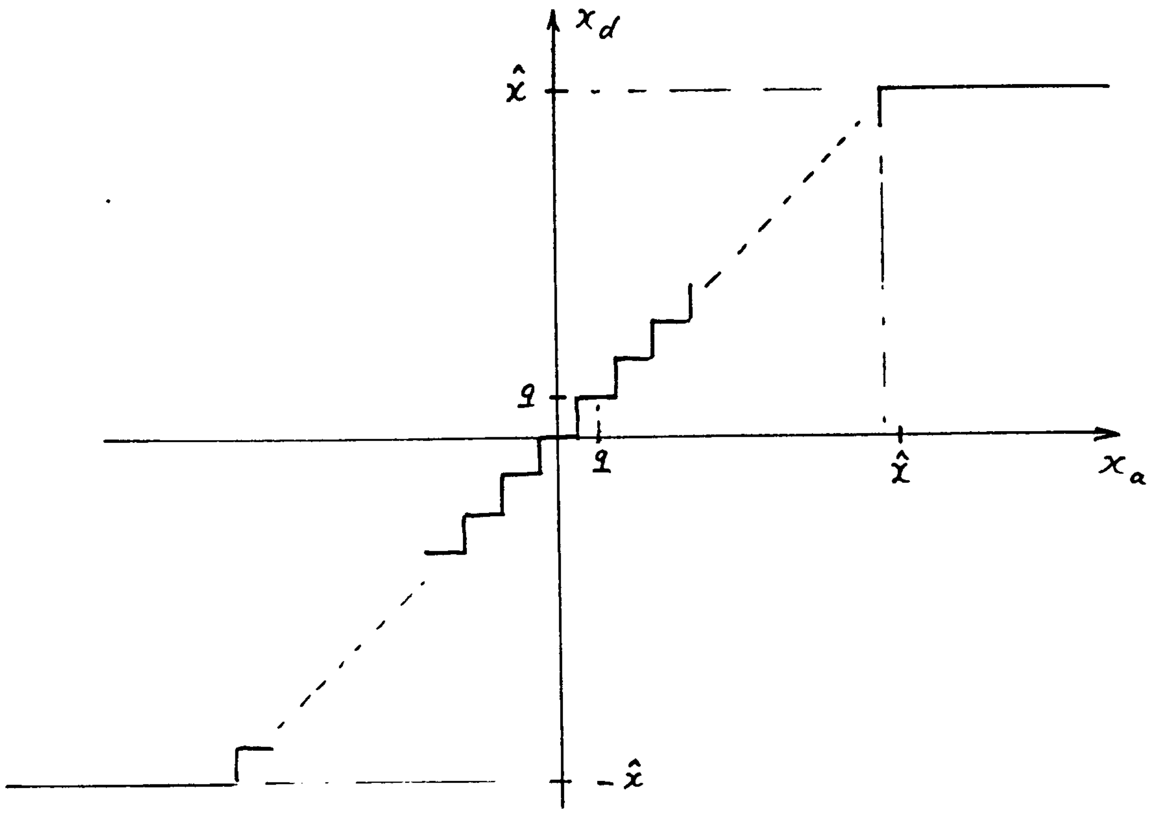

Fig.2.5.

This type of nonlinearity is frequently encountered in digital systems. A discussion of its influence on the signals will be deferred until we have discussed digital systems in more detail (see appendix A).
We will now concentrate on the generation of the discrete signal x(n). The delay in fig.2.5 accounts for the finite time that the comparator circuit requires for making the decisions. If the hold circuit is sufficiently accurate it has no influence on the sample values x(n). Moreover, it can be remarked from fig.2.3 that there is no difference if $x_a(t)$ rather then $y_a(t)$ is applied to the A/D converter since this is an idealized element that operates instantaneously. There is thus no need to compute the spectrum of $y_a(t)$ (which is difficult even if $\tau \approx T$) to find the output spectrum at the A/D converter.

For the analysis of a practical A/D converter the sampling and holding does therefore not play a role, and the practical A/D converter may thus just as well be modeled by merely an ideal A/D converter and a quantizer

2.4. Digital to analog converter

A digital to analog converter will be represented by the symbol of fig. 2.6.

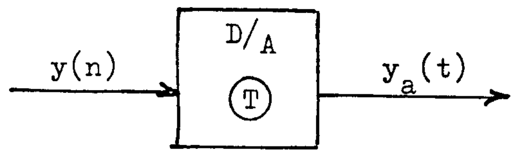

Fig.2.6

Its operation is defined by:

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(n)\,\delta_a(t-nT) \qquad (2.$$

and similarly as the A/D converter it is characterized by a single parameter T.

From (2.10) the spectrum can easily be derived:

$$Y_a(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y(n)\,\delta_a(t-nT)\; e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} y(n) \int_{-\infty}^{\infty} \delta_a(t-nT)\; e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} y(n)\; e^{-jn\omega T}$$

Therefore

$$Y_a(\omega) = Y(\omega T). \qquad (2.11)$$

Eq.(2.11) should be interpreted with some care.
Although the spectra of the two signals are equal if we set $\theta = \omega T$ there remains an important difference. To obtain the digital signal y(n) from its spectral representation $Y(\theta)$ we must integrate over a finite interval of length 2π (see eq. 1.12)

$y_a(t)$ can only be obtained from $Y_a(\omega)$ if we integrate over the whole frequency interval (see eq.(2.2))

2.5. A practical D/A converter

In practice a D/A converter will of course never produce a sequence of δ-functions. More realistic is it to assume that it produces pulses $p_a(t)$ of a form as shown in fig.2.7, with corresponding spectrum, $P_a(\omega)$ (fig.2.8) that are weighted with the sampling values of the digital signal.

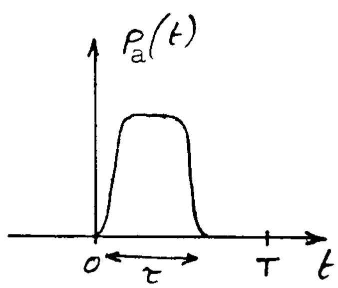

Fig.2.7

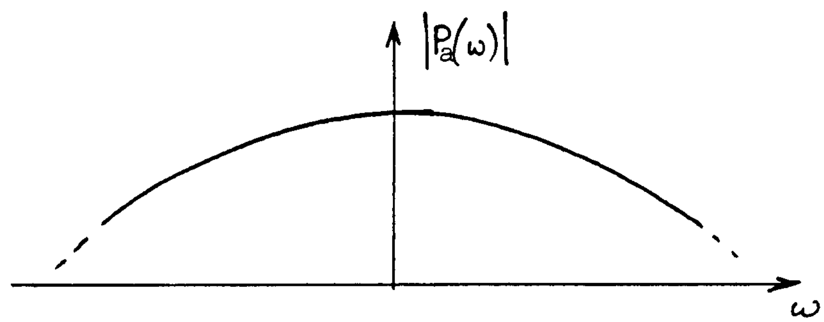

Fig.2.8

Thus the output signal will be:

$$\tilde{y}_a(t) = \sum_{n=-\infty}^{\infty} y(n) \,.\, p_a(t-nT). \tag{2.12}$$

From the identity

$$p_a(t) = p_a(t) * \delta_a(t) \tag{2.13}$$

it follows that

$$\tilde{y}_a(t) = y_a(t) * p_a(t) \tag{2.14}$$

and thus

$$\tilde{Y}_a(\omega) = Y_a(\omega) \,.\, P_a(\omega) \tag{2.15}$$

which means that $\tilde{y}_a(t)$ can be interpreted as the output of an analog filter with impulse response $p_a(t)$ to which $y_a(t)$ is applied as input signal. The situation is indicated in fig.2.9, and the corresponding spectra are displayed in fig.2.10. (Note that of the digital signal y(n) only the fundamental interval is shown according to the convention that we have adopted).

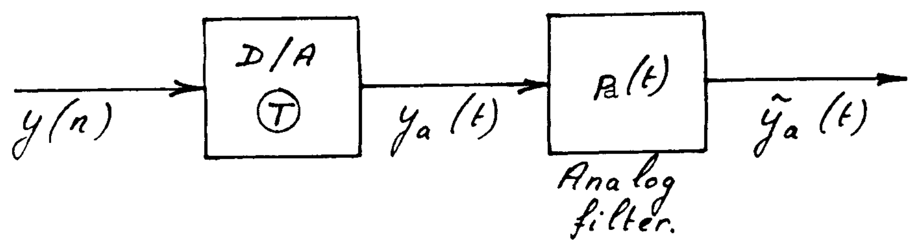

Fig.2.9.

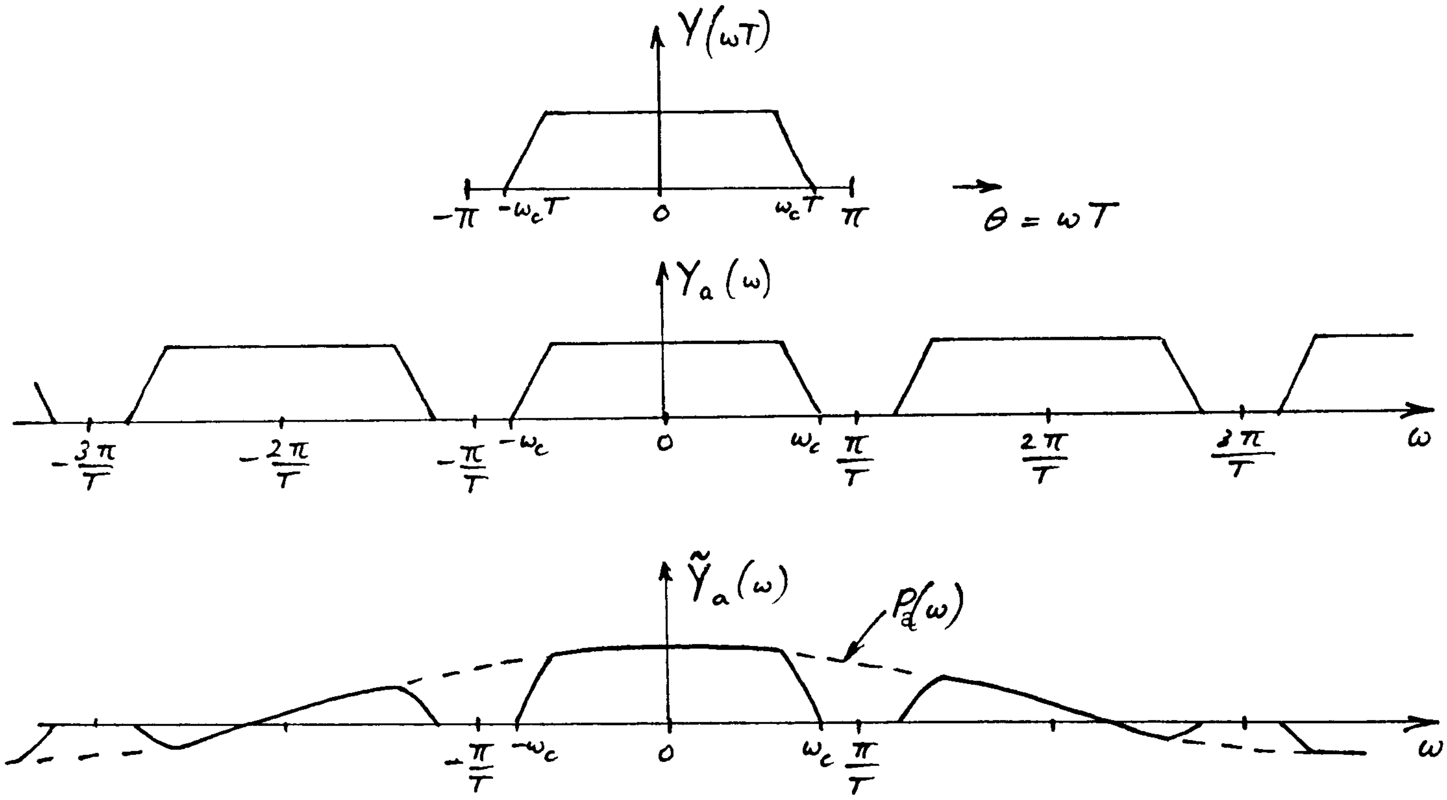

Fig.2.10

In general it will be desirable to obtain an output signal with a spectrum that resembles $Y(\omega T)$ as much as possible for $|\omega| < \frac{\pi}{T}$ and is zero elsewhere. In fact if we could obtain this goal, i.e.

$$Y_a(\omega) = \begin{cases} T.Y(\omega T) & |\omega| \leq \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases} \tag{2.16}$$

then we would possess the ideal D/A converter, because the cascade of an A/D and such a D/A converter would be like a mere through connection for bandlimited signals. This follows from eq-(2.10) and (2.16).

From eq. (2.15) we can conclude that (2.16) can only be satisfied if $P_a(\omega)$ has an "ideal low-pass characteristic":

$$P_a(\omega) = \begin{cases} T & |\omega| < \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases} \tag{2.17}$$

which means that the D/A converter must produce pulses of the form:

$$p_a(t) = \frac{\sin \frac{\pi}{T} t}{\pi t/T} \tag{2.18}$$

These pulses extend from $-\infty$ to ∞ and can of course not be produced in an actual system. But this derivation may serve as a constructive proof of the sampling theorem which states that an analog signal which is bandlimited to the frequency $\omega_c = \frac{\pi}{T}$ may be represented by samples taken at instants not farther apart than T. (see Signal Analysis, section 5).

In practice the situation mostly is less severe than eq.(2.16) suggests. Firstly some over-sampling is often used which means that $T < \frac{\pi}{\omega_c}$ as in fig.2.10 and this gives a margin of 2. $(\frac{\pi}{T} - \omega_c)$ between passband and stopband. Secondly the attenuation needs not be infinite over the whole stopband. Often an attenuation of say 40 or 60dB will suffice. Thirdly, if a filter is necessary anyway to attenuate the so-called higher order spectral components then this filter may as well be used to compensate the deformation of the spectrum in the passband due to the pulse form. A filter with transmission function $H(\omega)$ as shown in fig.2.11 placed in cascade with the D/A converter with pulse form $p_a(t)$ used in fig.2.10 will produce an output spectrum as shown in fig.2.12. This means that in practice a D/A converter with relatively smooth pulses may be used if it is followed by a (realizable) filter that has a transmission function that compensates for passband distortion and that gives sufficient stopband attenuation.

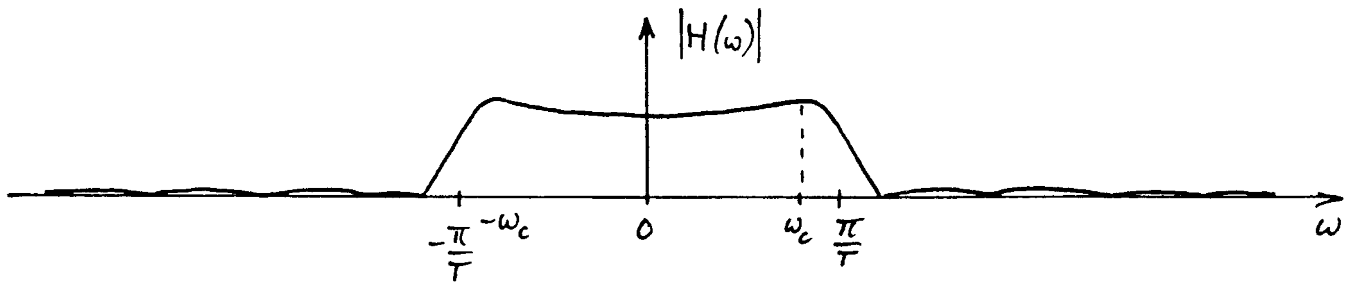

Fig.2.11.

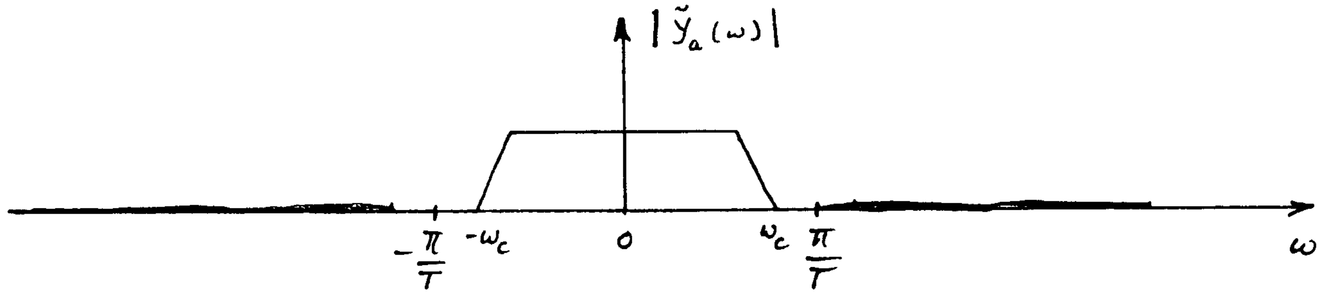

Fig. 2.12.