DETECTION THEORY AND BAYES CRITERION

RECONOCIMIENTO DE PATRONES

LABORATORIO AVANZADO DE PROCESAMIENTO DE IMÁGENES

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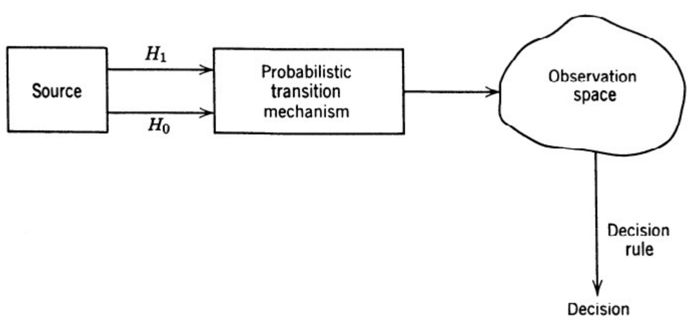
Binary detection, two hypotheses



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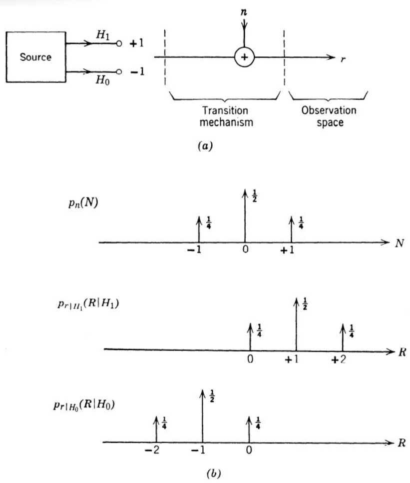
Example



$$H_1: r = 1 + n,$$
$$H_0: r = -1 + n.$$

N-dimensional observation space

$$\mathbf{r} \triangleq \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$$

Possible decisions

1. H_0 true; choose H_0.
2. H_0 true; choose H_1.
3. H_1 true; choose H_1.
4. H_1 true; choose H_0.

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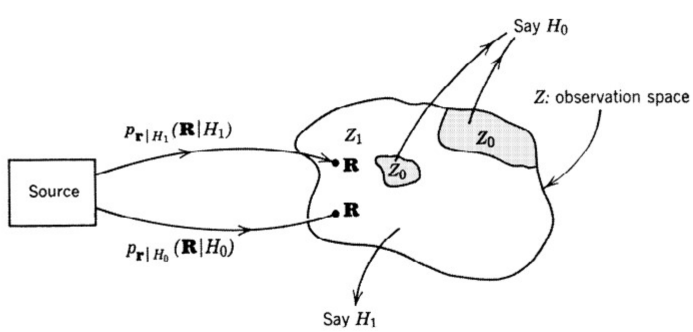
Bayes Criterion

$$\begin{aligned}\mathcal{R} = & \; C_{00}P_0 \Pr(\text{say } H_0|H_0 \text{ is true}) \\ & + C_{10}P_0 \Pr(\text{say } H_1|H_0 \text{ is true}) \\ & + C_{11}P_1 \Pr(\text{say } H_1|H_1 \text{ is true}) \\ & + C_{01}P_1 \Pr(\text{say } H_0|H_1 \text{ is true}).\end{aligned}$$

Bayes test:
- A priori probabilities
- Associated cost
- Risk: expected value of cost



Decision regions

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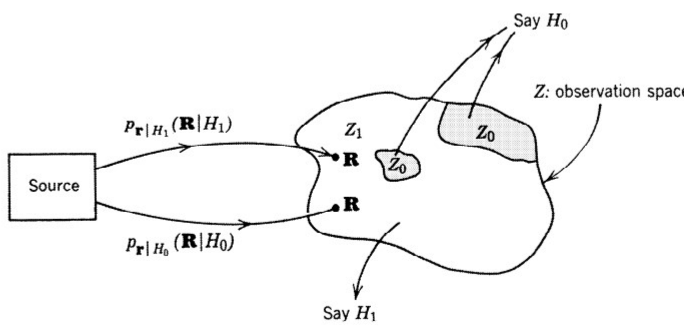
Decision regions

$$\mathcal{R} = C_{00}P_0 \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + C_{10}P_0 \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + C_{11}P_1 \int_{Z_1} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} + C_{01}P_1 \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}.$$



$$\mathcal{R} = P_0C_{00} \int_{Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_0C_{10} \int_{Z-Z_0} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} + P_1C_{01} \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} + P_1C_{11} \int_{Z-Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R}.$$

Wrong decisions costs higher tan correct decisión costs:

$$C_{10} > C_{00}, \quad C_{01} > C_{11}.$$

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Observing that:

$$\int_Z p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R} = \int_Z p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} = 1,$$

Risk reduces to

$$\mathcal{R} = P_0C_{10} + P_1C_{11} + \int_{Z_0} \{[P_1(C_{01} - C_{11})p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)] - [P_0(C_{10} - C_{00})p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)]\}\, d\mathbf{R}.$$

Remember:

$$C_{10} > C_{00}, \quad C_{01} > C_{11}.$$

All values of **R** where the second term is larger than the first should be included in $\mathbf{Z_0}$ because they contribute a negative amount.
All values of **R** where the first term is larger than the second sholud be excluded from $\mathbf{Z_0}$ (assigned to $\mathbf{Z_1}$) because they contribute a positive amount to the integral.

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If $P_1(C_{01} - C_{11})p_{\mathbf{r}|H_1}(\mathbf{R}|H_1) \geq P_0(C_{10} - C_{00})p_{\mathbf{r}|H_0}(\mathbf{R}|H_0),$ Assign **R** to H_1

Alternately

$$\frac{p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})}.$$

Likelihood ratio

$$\boxed{\Lambda(\mathbf{R}) \triangleq \frac{p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)}.}$$

Threshold

$$\eta \triangleq \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})}.$$

Likelihood ratio test

$$\Lambda(\mathbf{R}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta.$$

Or

$$\ln \Lambda(\mathbf{R}) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta.$$

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Example

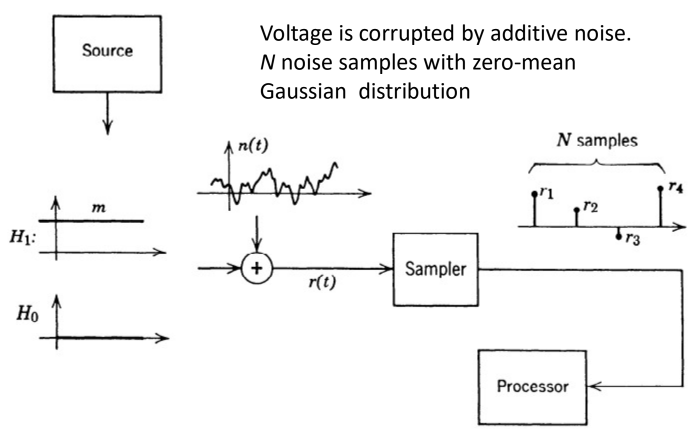
$$H_1: r_i = m + n_i \qquad i = 1, 2, \dots, N,$$
$$H_0: r_i = \phantom{m + {}} n_i \qquad i = 1, 2, \dots, N,$$

$$p_{n_i}(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{X^2}{2\sigma^2}\right),$$

$$p_{r_i|H_1}(R_i|H_1) = p_{n_i}(R_i - m) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)$$

$$p_{r_i|H_0}(R_i|H_0) = p_{n_i}(R_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right).$$

Voltage is corrupted by additive noise.
N noise samples with zero-mean Gaussian distribution



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Samples statistically independent, therefore joint pdfs:

$$p_{\mathbf{r}|H_1}(\mathbf{R}|H_1) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right),$$

$$p_{\mathbf{r}|H_0}(\mathbf{R}|H_0) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right).$$

Likelihood ratio

$$\Lambda(\mathbf{R}) = \frac{\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)}{\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{R_i^2}{2\sigma^2}\right)}.$$

Or

$$\ln \Lambda(\mathbf{R}) = \frac{m}{\sigma^2} \sum_{i=1}^{N} R_i - \frac{Nm^2}{2\sigma^2}.$$

Likelihood ratio test

$$\frac{m}{\sigma^2} \sum_{i=1}^{N} R_i - \frac{Nm^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

$$\sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m} \ln \eta + \frac{Nm}{2} \triangleq \gamma.$$

Sufficient statistic

$$\sum_{i=1}^{N} R_i$$

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Example 2

N independent values: r_1, r_2, r_3, ….., r_N, zero-mean Gaussian random variables with different variances.

$$p_{\mathbf{r}|H_1}(\mathbf{R}|H_1) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left(-\frac{R_i^2}{2\sigma_1^2}\right)$$

$$p_{\mathbf{r}|H_0}(\mathbf{R}|H_0) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left(-\frac{R_i^2}{2\sigma_0^2}\right).$$

$$\boxed{\Lambda(\mathbf{R}) \triangleq \frac{p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)}{p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)}.}$$

Substituting and taking logarithm

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \sum_{i=1}^{N} R_i^2 + N \ln \frac{\sigma_0}{\sigma_1} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta.$$

Sufficient statistic $l(\mathbf{R}) = \sum_{i=1}^{N} R_i^2,$

Equivalent test for $\sigma_1^2 > \sigma_0^2$

$$l(\mathbf{R}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}\left(\ln \eta - N \ln \frac{\sigma_0}{\sigma_1}\right) \triangleq \gamma.$$

Equivalent test for $\sigma_1^2 < \sigma_0^2$

$$l(\mathbf{R}) \underset{H_1}{\overset{H_0}{\gtrless}} \frac{2\sigma_0^2\sigma_1^2}{\sigma_0^2 - \sigma_1^2}\left(N \ln \frac{\sigma_0}{\sigma_1} - \ln \eta\right) \triangleq \gamma'$$

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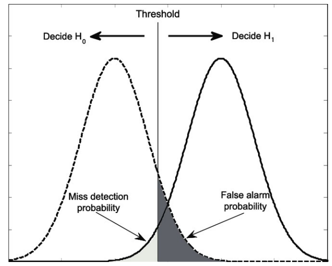
Decision regions in the Bayes test

False alarm $P_F = \int_{Z_1} p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)\, d\mathbf{R},$

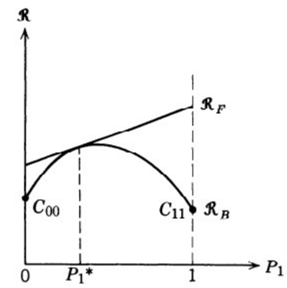
Detection $P_D = \int_{Z_1} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R},$

Miss $P_M = \int_{Z_0} p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)\, d\mathbf{R} = 1 - P_D.$



$$\mathcal{R} = P_0C_{10} + P_1C_{11} + \int_{Z_0} \{[P_1(C_{01} - C_{11})p_{\mathbf{r}|H_1}(\mathbf{R}|H_1)] - [P_0(C_{10} - C_{00})p_{\mathbf{r}|H_0}(\mathbf{R}|H_0)]\}\, d\mathbf{R}.$$ (slide 6)

$$\mathcal{R} = P_0C_{10} + P_1C_{11} + P_1(C_{01} - C_{11})P_M - P_0(C_{10} - C_{00})(1 - P_F).$$



- $\mathcal{R}_B$ is the true Bayes Risk for varying P_1
- $\mathcal{R}_F$ is the Bayes test for fixed P_M and P_F., i.e. the risk grows if true P_1 is unknown.

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Minimax test

$$\mathcal{R} = P_0C_{10} + P_1C_{11} + P_1(C_{01} - C_{11})P_M - P_0(C_{10} - C_{00})(1 - P_F).$$

- Minimize the maximum posible risk
- Derive R with respecto to P1 and equal to zero:

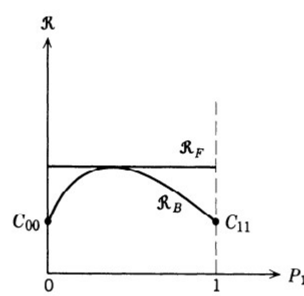
$$(C_{11} - C_{00}) + (C_{01} - C_{11})P_M - (C_{10} - C_{00})P_F = 0.$$

To guarantee concave curve, we assume $C_{00} = C_{11} = 0$

and denote $C_{01} = C_M,$ $C_{10} = C_F.$

Rewriting the Risk: $\mathcal{R}_F = C_FP_F + P_1(C_MP_M - C_FP_F) = P_0C_FP_F + P_1C_MP_M$

Minimax equation: $C_MP_M = C_FP_F.$



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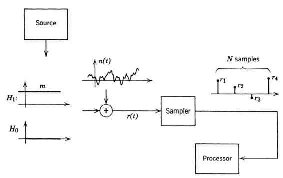
Receiver Operating Characteristic

- How to evaluate the performance of a likelihood test?
- Bayes Risk depends on P_F and P_D
- Recall Example 1



$$p_{r_i|H_1}(R_i|H_1) = p_{n_i}(R_i - m) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(R_i - m)^2}{2\sigma^2}\right)$$

$$p_{r_i|H_0}(R_i|H_0) = p_{n_i}(R_i) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{R_i^2}{2\sigma^2}\right).$$

Likelihood ratio test

$$\frac{m}{\sigma^2}\sum_{i=1}^{N} R_i - \frac{Nm^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta$$

$$\sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{m}\ln \eta + \frac{Nm}{2} \triangleq \gamma.$$

Rearranging

$$l = \frac{1}{\sqrt{N}\,\sigma}\sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma}{\sqrt{N}\,m}\ln \eta + \frac{\sqrt{N}\,m}{2\sigma}.$$

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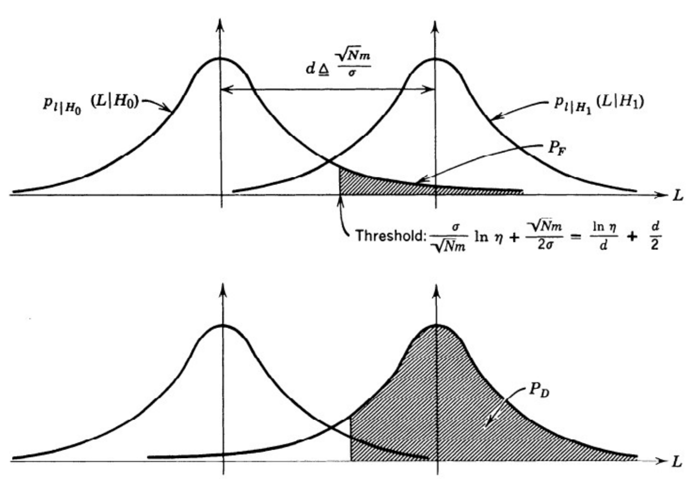
Error probabilities

$$l = \frac{1}{\sqrt{N}\,\sigma}\sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma}{\sqrt{N}\,m}\ln \eta + \frac{\sqrt{N}\,m}{2\sigma}.$$



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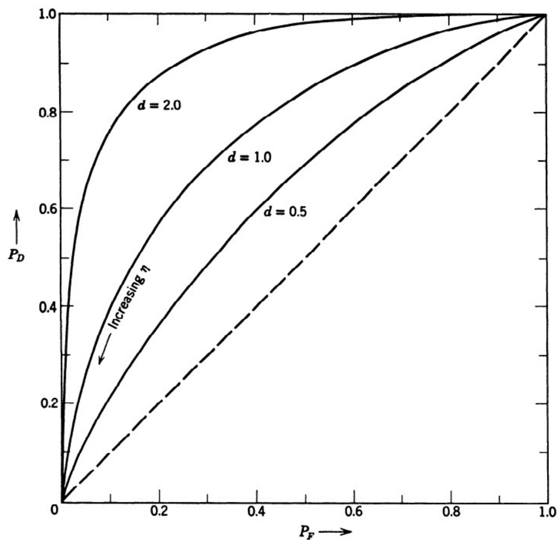
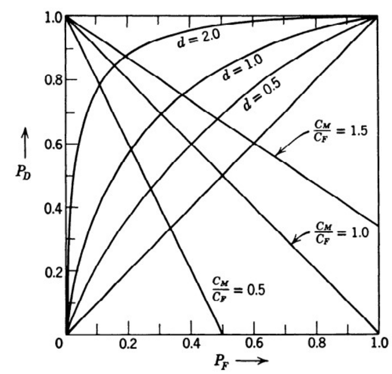
ROC: Gaussian variables with different means



Properties:

- ROCs are concave, above $P_D = P_F$
- Slope of ROC is equal to threshold η
- Minimax operating point is the intersection with line

$$C_F P_F = C_M P_M = C_M(1 - P_D),$$



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Homework random processes

Show that sinusoidal wave with random phase

$$X(t) = A \cos(\omega t + \Theta)$$

with pase Θ uniformly distributed on [0, 2π] is stationary and ergodic

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