5. Change of the sampling rate.

Thus far we have shown how an analog signal can be converted into a digital signal, how such a digital signal can be filtered in a digital filter and modulated by multiplying its samples with a digital carrier, and finally, if desired converted into an analog signal again.
Of utmost importance in this whole procedure was the fact that in the digital part of such a signal processing scheme only frequencies in the range $0 \leq \omega < \frac{\pi}{T}$ can be distinguished where $T = 1/f_s$ is the sampling period. The consequence hereof was that before A/D conversion we had to make sure that the analog signal was bandlimited to this interval, since otherwise distortion due to aliasing occurs. If the sampling frequency is fixed this often means that an analog prefiltering must take place. If we can choose the sampling frequency then we can select it sufficiently large so that the aliasing is negligible. But then do we have to keep the sampling frequency so high in the whole digital system ?

Consider the circuit in fig.5.1, in which an analog signal $x_a(t)$ is A/D converted with a frequency $f_s = 1/T$ and the resulting digital signal $x(n)$ is subsequently filtered with a lowpass filter which attenuates all frequencies in the range $\omega_c T < \theta < \pi$.
The output signal of the filter thus has all its energy in the band $|\theta| < \omega_c T$ and according to the sampling theorem, such a signal may be represented by $2f_c$ samples per second rather then the f_s samples by which it is represented now.

Especially if $2f_c \ll f_s$ it seems to be advantageous to lower the sampling frequency to a value close to $2f_c$.

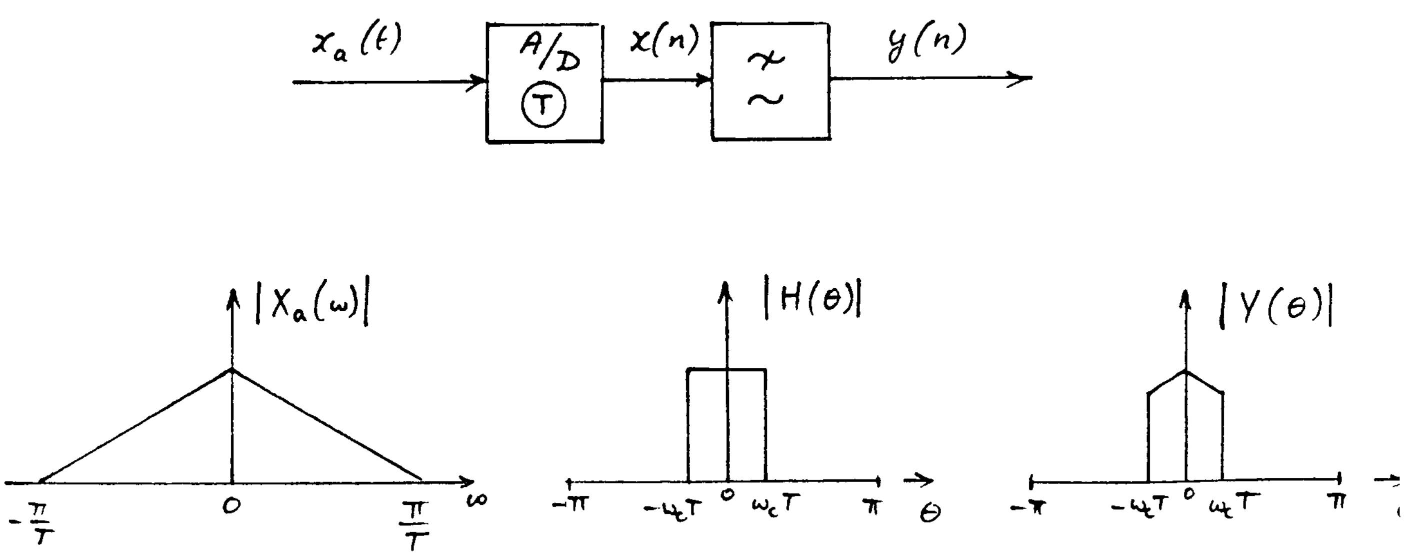


fig.5.1.

An other example where a change of the sampling frequency may be advantageous is shown in fig.5.2. Here an analog signal $x_a(t)$ band-limited to $|\omega| < \omega_0$ is converted to a digital signal x(n) with sampling frequency $f_s = 1/T$. This digital signal is then filtered in a bandpass filter the output of which modulates a carrier of frequency f_c.

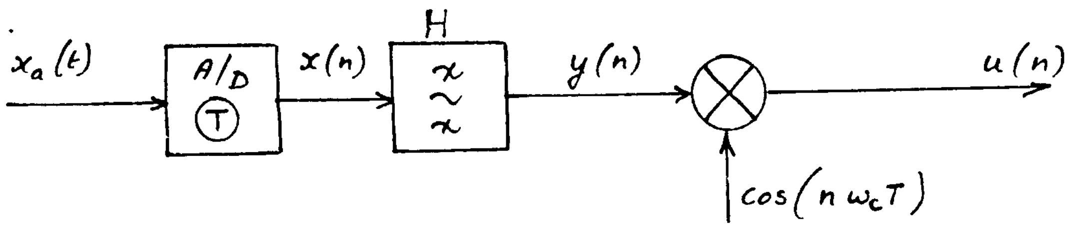


fig.5.2.

The spectra of the various signals are shown in fig.5.3. From these spectra it can be seen that to represent the signal u(n) we must have $(\omega_c + \omega_2)T < \pi$ or a sampling frequency f_s according to

$$f_s = \frac{1}{T} > \frac{\omega_c + \omega_2}{\pi} = 2(f_c + f_2).$$

On the other hand the signals x(n) and y(n) are bandlimited to the range $|\theta| < \omega_0 T$ which means that they could be represented by $2f_0 = \omega_0/\pi$ samples per second. But then before the modulation the sampling rate must be increased.

In this section we will introduce two boxes that perform a sampling rate decrease (SRD) and increase (SRI) respectively. It should be noted, however, that in an actual implementation of a digital system in which sampling rate increase or decrease occurs, such boxes can hardly ever be recognized as separate entities. In any realization of digital systems it is worthwhile to try to combine functions of the various elements whenever possible. In particular with the SRD and SRI this is often quite simple, as will be shown for example in part II where an interpolating filter will be discussed. The introduction of the SRD and SRI as separate elements has proved to be of considerable help in the analysis of such systems, however.

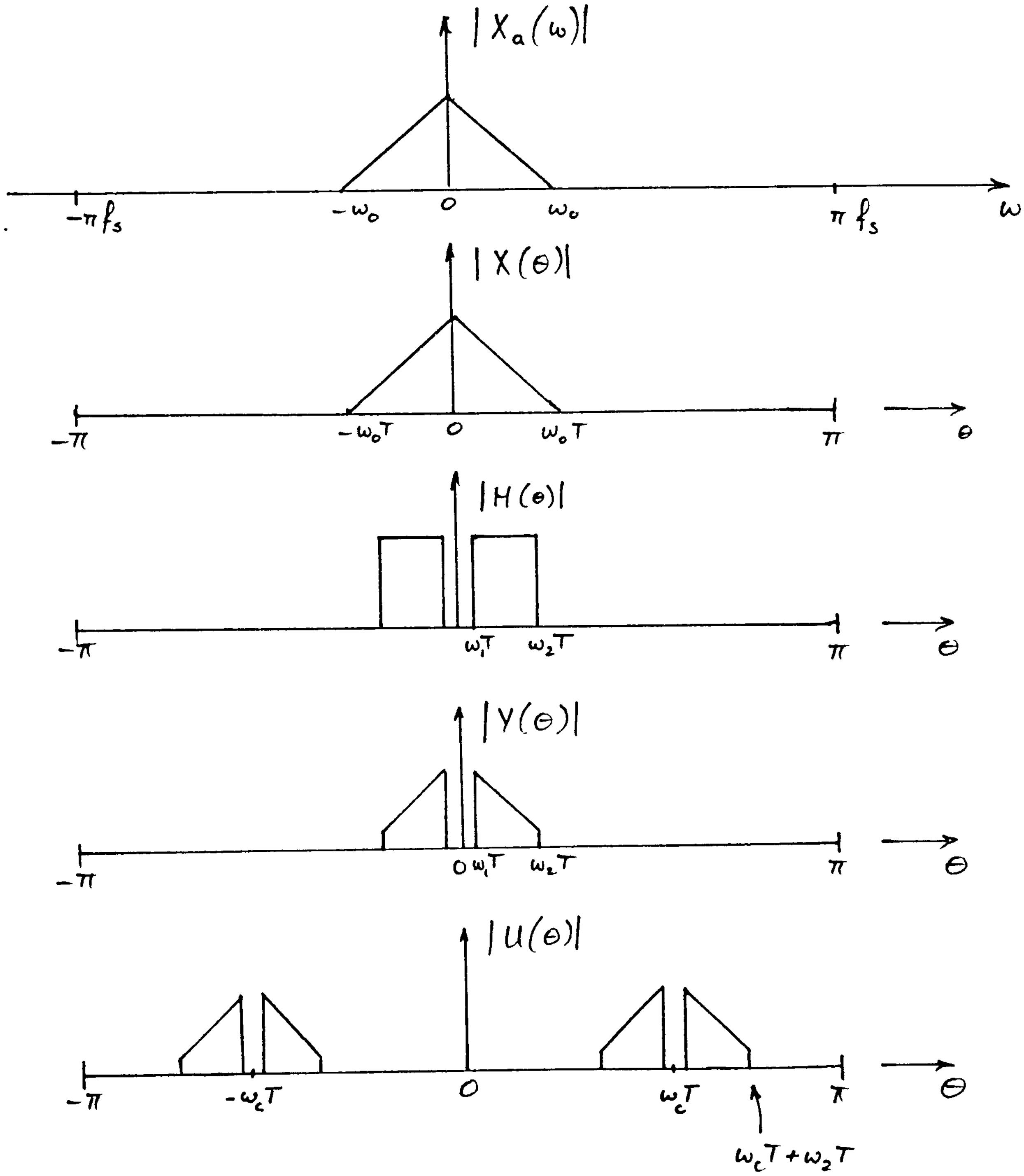

|X_a(ω)|
-π f_s
-ω_0
0
ω_0
π f_s
ω
|X(θ)|
-π
-ω_0T
0
ω_0T
π
θ
|H(θ)|
-π
ω_1T
ω_2T
π
θ
|Y(θ)|
-π
0 ω_1T
ω_2T
π
θ
|U(θ)|
-π
-ω_cT
0
ω_cT
π
θ
ω_cT+ω_2T

fig.5.3.

5.1. Sampling rate decrease.

The sampling rate decrease (SRD) by an integer factor N will be represented by the symbol depicted in fig.5.4 and is described by:

$$y(n) = x(nN) \tag{5.1}$$

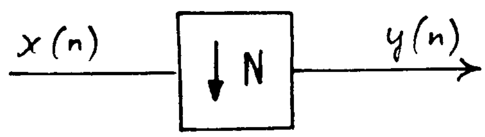


fig.5.4.

The output signal is thus obtained by taking every N'th sample of the input signal. An example with N=3 is given in fig.5.5, where for convenience the n-axis is scaled such that the time scales of the input and output correspond. If the sampling period of x(n) is T, then the sampling period of y(n) is N.T. and thus the sampling frequency is decreased by a factor N.

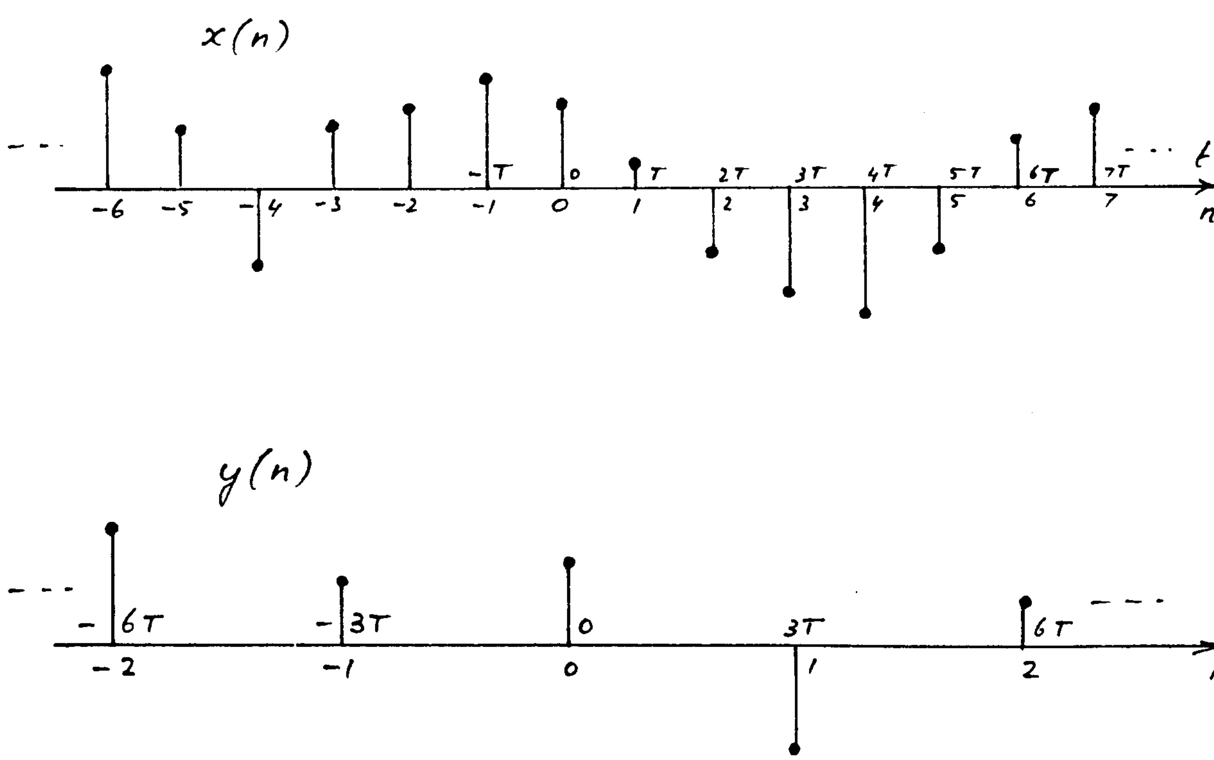


fig.5.5.

Now we will determine the relation between the spectra of x(n) and y(n), which can best be done by considering the fundamental interval $(0,2\pi)$ instead of $(-\pi,\pi)$.
We have

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta_x)\, e^{jn\theta_x}\, d\theta_x = \frac{1}{2\pi}\int_{0}^{2\pi} X(\theta_x)\, e^{jn\theta_x}\, d\theta_x$$

thus

$$x(nN) = \frac{1}{2\pi} \int_{0}^{2\pi} X(\theta_x)\, e^{jnN\theta_x}\, d\theta_x$$

$$= \frac{1}{2\pi} \int_{0}^{N.2\pi} X\left(\frac{N\theta_x}{N}\right) e^{jn(N\theta_x)}\, \frac{d(N\theta_x)}{N}$$

$$= \frac{1}{2\pi}\, \frac{1}{N} \int_{0}^{N.2\pi} X\left(\frac{\varphi}{N}\right)\, e^{jn\varphi}\, d\varphi$$

$$= \frac{1}{2\pi}\, \frac{1}{N} \sum_{k=0}^{N-1} \int_{k.2\pi}^{(k+1)2\pi} X\left(\frac{\varphi}{N}\right)\, e^{jn\varphi}\, d\varphi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{\varphi+k.2\pi}{N}\right)\, e^{jn(\varphi+k.2\pi)}\, d\varphi$$

And since

$$x(nN)=y(n)= \frac{1}{2\pi} \int_{0}^{2\pi} Y(\theta_y)\, e^{jn\theta_y}\, d\theta_y$$

we conclude:

$$Y(\theta_y) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{\theta_y + k.2\pi}{N}\right), \quad 0 \leq \theta_y \leq 2\pi. \qquad (5.2)$$

Similarly as with the A/D converter we see that parts of the spectrum of the input signal are folded to the fundamental interval of the output spectrum. The number of folding terms in this case is equal to (N-1) and thus finite. A pictorial interpretation of eq.(5.2) is given in fig.5.6 for N=3, where for convenience the spectra are displayed both for the conventional fundamental interval $(-\pi,\pi)$ and for $(0,2\pi)$.

In section 2 on A/D conversion it has been indicated that the relati - frequency θ and the normal frequency ω were related by $\theta=\omega T$ where T is the sampling rate. Since the sampling period T_x of x(n) is equal to T and that of y(n) is $T_y = N.T$ we have

$$\theta_x = \omega T_x = \omega T \qquad (5.3)$$

$$\theta_y = \omega T_y = N.\omega.T \qquad (5.4)$$

If we want the same frequency information for the spectra of x and y then a scaling of the relative frequency axis according to eq.(5.3) and (5.4) must be used, as is done in fig.5.6. In this figure the spectrum of x is divided into six different parts, each having its own shading.
Spectral components for $0 \leq \theta_x < \pi$ have been labeled from 1 to 3, and the corresponding components at negative frequencies 1^* to 3^* to accentuate the fact that for real x(n) the spectrum will satisfy $X(-\theta) = X^*(\theta)$.

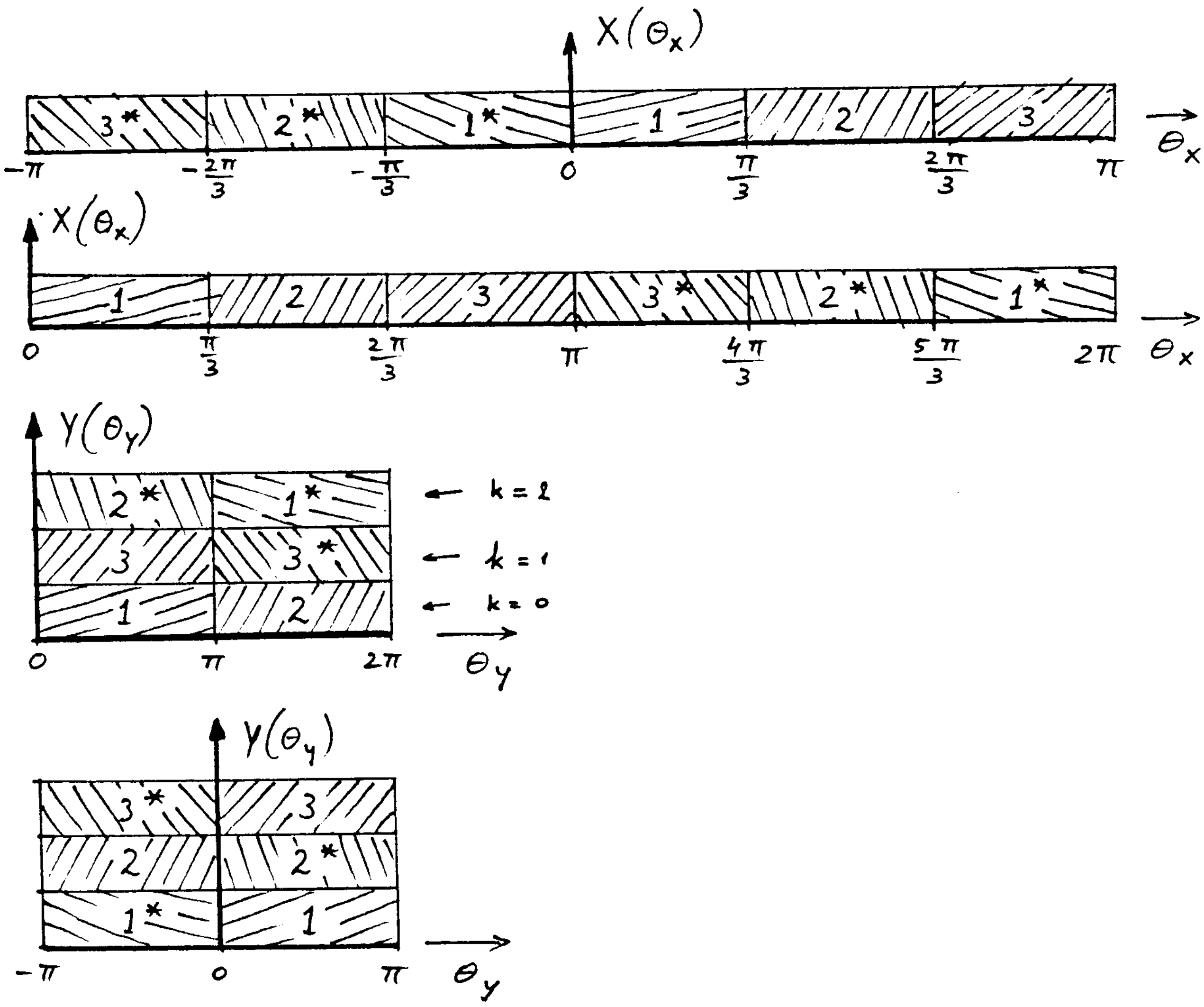


fig.5.6.

In general the components labeled $2, 2^*$, 3 and 3^* will be undesired, and these terms are again designated as <u>aliasing terms</u>. Aliasing can therefore only be avoided if the spectrum of x(n) is bandlimited to half of the output rate $\pi/T_y = \pi/NT_x$; thus

$$X(\theta_x) = 0 \qquad \frac{\pi}{N} \leqslant |\theta_x| < \pi$$

in which case

$$Y(\theta_y) = \frac{1}{N} \cdot X\left(\frac{\theta_y}{N}\right) \qquad -\pi < \theta_y \leq \pi$$

or

$$Y(\omega T_y) = \frac{1}{N} \cdot X(\omega T_x) \qquad \frac{-\pi}{T_y} < \omega \leq \frac{\pi}{T_y} \qquad (5.5)$$

5.2. Sampling rate increase.

The sampling rate increase (SRI) with an integer factor N will be indicated by the symbol of fig.5.7 and is described by:

$$y(n) = \begin{cases} x(n/N) & n=0, \pm N. \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases} \qquad (5.6)$$

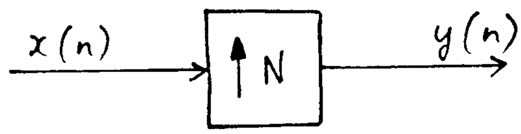


fig.5.7.

The interpretation of eq. (5.6) is that y(n) consists of the samples of x(n) with (N-1)-samples with value zero inserted in between any two original samples. This is shown in fig.5.8 for N=3 were again the axes are scaled such that the time axis in both cases correspond. For this to be the case we must set:

$$T_x = N. T_y \qquad (5.7)$$

which means that the sampling frequency is increased by the factor N.
The spectral representation of the SRI follows from:

$$Y(\theta_y) = \sum_{n=-\infty}^{\infty} y(n) \; e^{-j\theta_y . n}$$

$$= \sum_{n=0,\pm N, \dots} x(n/N) \; e^{-j\theta_y . n}$$

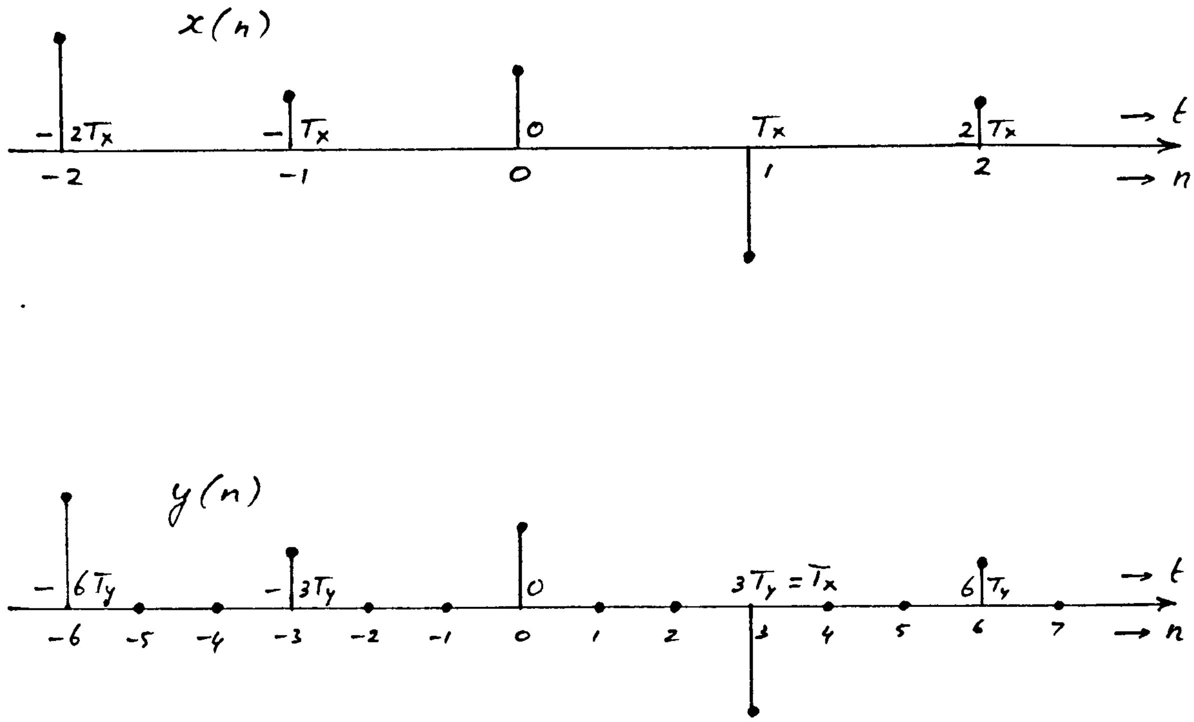


fig.5.8.

Substituting k = n/N we get

$$Y\ (\theta_y)\ =\ \sum_{k=-\infty}^{\infty} x(k)\ e^{-j\theta_y . N . k}$$

$$Y\ (\theta_y)\ =\ X\ (N\theta_y) \tag{5.8}$$

As before we must introduce the corresponding sampling periods to obtain the actual frequencies in both cases :

$$\theta_y\ =\ \omega T_y \tag{5.9}$$

$$\theta_x\ =\ \omega T_x\ =\ N.\omega T_y \tag{5.10}$$

Inserting (5.9) and (5.10) into (5.8) gives:

$$Y\ (\omega T_y)\ =\ X\ (\omega T_x) \tag{5.11}$$

Tne interpretation of eq. (5.8) or (5.11) is that a fundamental interval of the spectrum of y consists of N intervals of the spectrum of x, which means N times repeated the fundamental interval of X(θ) because X(θ) is periodic. This is illustrated in fig.5.9 for N=3.

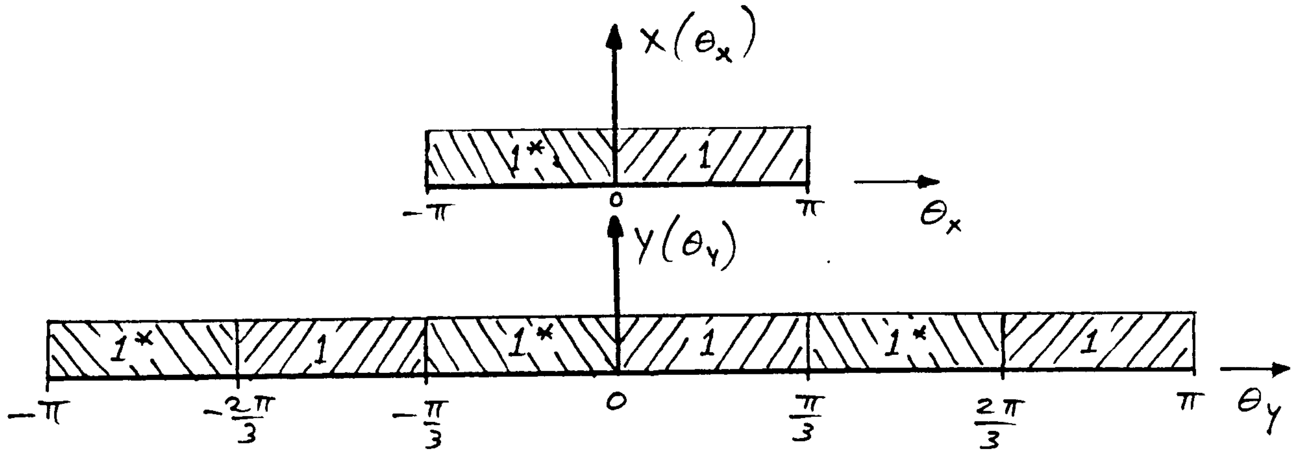


fig.5.9.

5.3. Interpolator.

In the application of an SRI that was discussed in the introduction of this section the desired spectral relation was not the one given by eq. (5.11). Rather what we would like is that the spectrum of y equals that of x on the interval $|\omega| < \pi / T_x$ and is zero outside this interval. Thus:

$$Y(\omega T_y) = \begin{cases} X(\omega T_x) & |\omega| < \frac{\pi}{T_x} \\ 0 & \frac{\pi}{T_x} \leqslant |\omega| < \frac{\pi}{T_y} \end{cases} \qquad (5.12)$$

From fig.5.9 it is easily seen that a signal with a spectrum given by (5.12) can be obtained by passing the output of the SRI through a low-pass filter with cut-off frequency $\omega_c = \pi/T_x$.

Now consider the system in fig.5.10. It consists of two parts, the first of which is an A/D converter with period T_1 the second an A/D converter with period T_2 followed by an SRI and lowpass filter with cut-off frequency π/T_2.

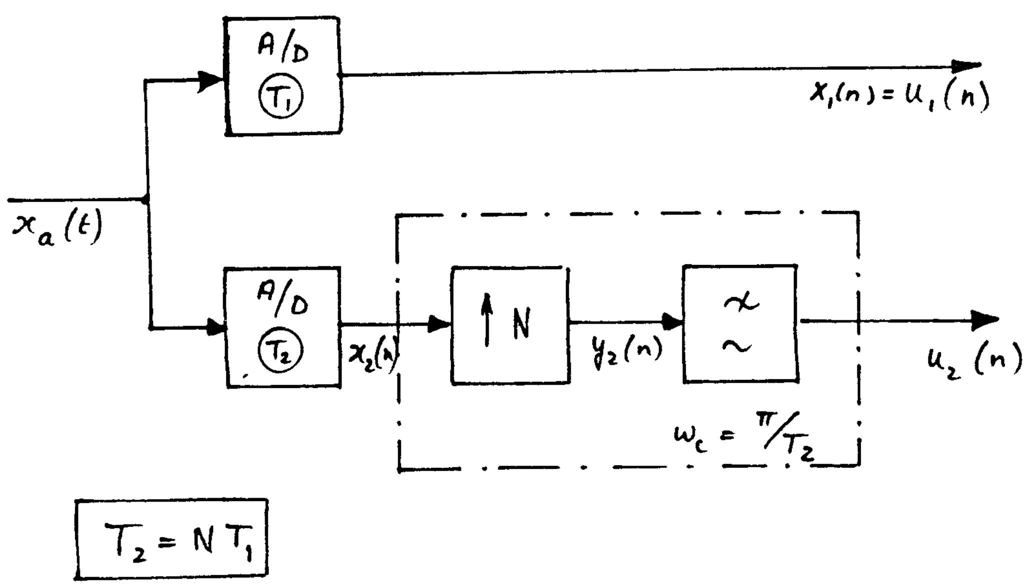
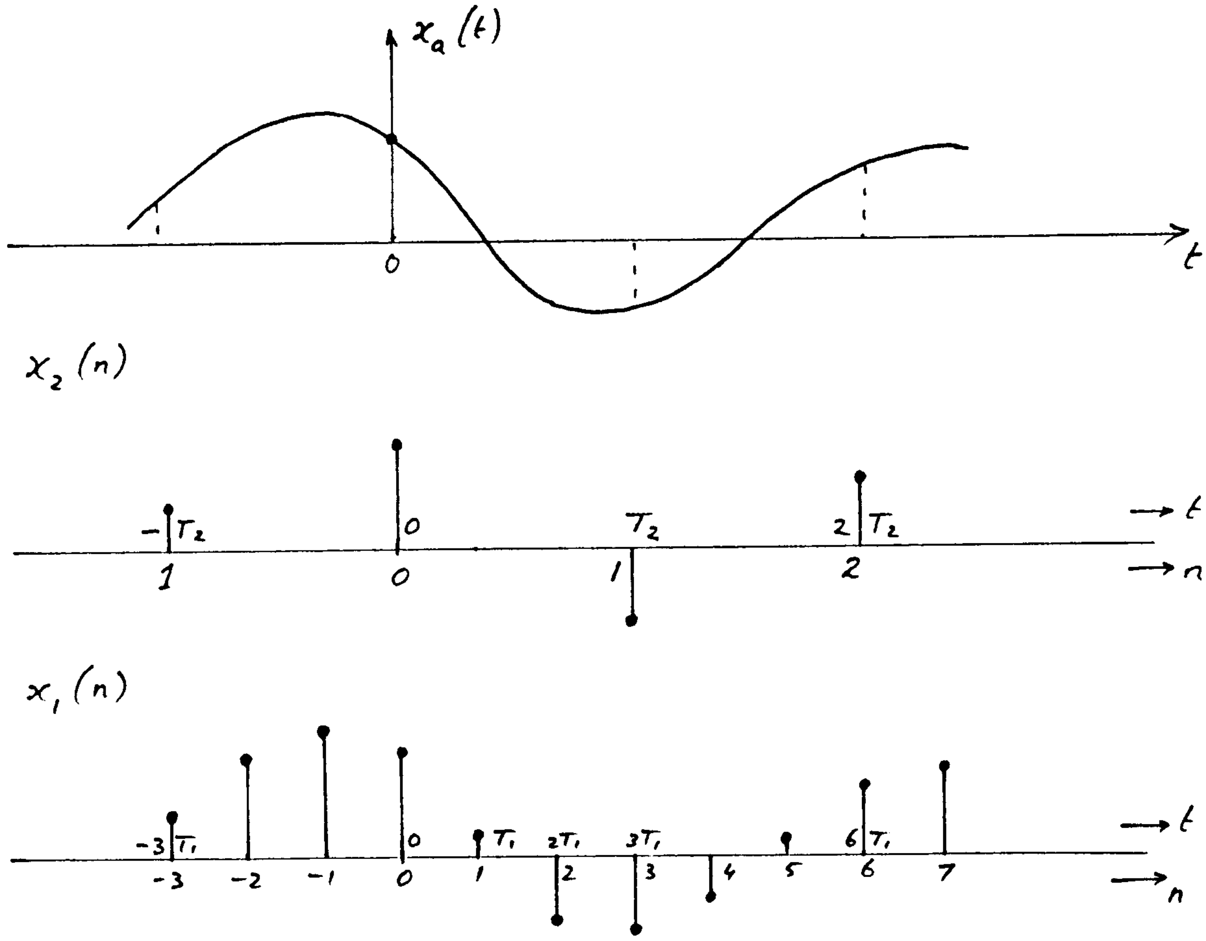


fig.5.10.

If $x_a(t)$ is bandlimited to π/T_2, then from the foregoing it follows that $u_1(n) \equiv u_2(n)$.

The combination of an SRI and the lowpass filter is often called an <u>interpolator</u> because with the input signal $x_2(n)$ it produces output samples $u_2(n)$ that correspond to the samples of the "original" analog signal $x_a(t)$ taken at the higher rate $1/T_1=N/T_2$, as illustrated in fig.5.11 for N=3.



<u>fig.5.11.</u>

5.4. Change of sampling frequency by a rational fraction.

We have shown now that a change of the sampling rate by an arbitrary integer factor can be accomplished by an SRD or a combination of an SRI and a lowpass filter. The latter combination has been given the name interpolator. To conclude this section we remark that a change in sampling rate by any rational fraction can be accomplished by a cascade of these two devices.

Let

$$T_y = \frac{N_2}{N_1} \cdot T_x$$

with N_1 and N_2 two arbitrary integers. This change in sampling rate can be obtained by the system of fig.5.12.

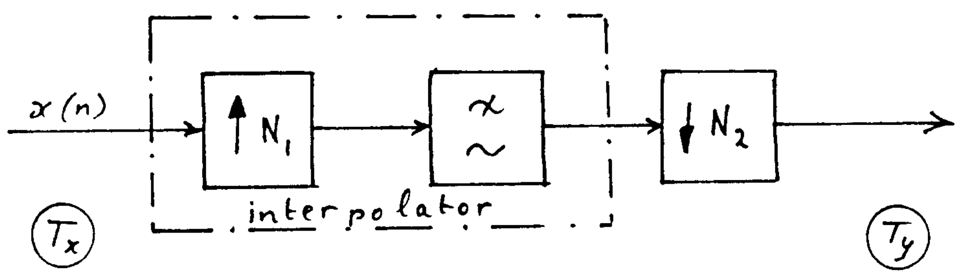


fig.5.12.

We distinguish two cases:

a) $N_1 > N_2$ which corresponds to an increase in sampling rate.
According to the previous derivation we have:

$$Y(\omega T_y) = \begin{cases} \frac{1}{N_2} X(\omega T_x) & 0 \leq |\omega| < \frac{\pi}{T_x} \\ 0 & \frac{\pi}{T_x} \leq |\omega| < \frac{\pi}{T_y} \end{cases}$$

b) $N_1 < N_2$ which corresponds to a decrease in sampling rate. In that case an undistorted output spectrum will only be obtained if x(n) was bandlimited to π/T_y in which case

$$Y(\omega T_y) = \frac{1}{N_2} X(\omega T_x) \qquad 0 \leq |\omega| < \frac{\pi}{T_y} .$$